► PIETRO BROCCI, *Proof-theoretic remarks on extensions of the Kripke-Feferman theory of truth*.
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Axiomatic theories of truth are extensions of an arithmetical base theory (which also plays the role of a theory of syntax) with a unary predicate for truth and axioms for it. In light of Tarski's undefinability result, they can be seen as a truth-theoretic formal framework where consistency is achieved not by defining truth in a meta-theory but where truth is kept internal to theory by weakening it in some ways.

We consider in particular the theory Kripke-Feferman (KF), which is an axiomatization of Kripke's semantic theory of truth based on strong-Kleene logic. The theory was first introduced by Feferman as a truth-theoretic framework to study transfinite iteration of reflection principles (see [4]). In particular, KF can be used to define the notion of reflective closure of a schematic axiom system. This iteration process reaches a proof-theoretic fixed-point at the Feferman-Schütte ordinal Γ_0, often understood as the limit of predicative mathematics.

It has been recently put to the test whether Γ_0 is a limit of predicative reasoning, or if this feature is just contingent on the way we define the ordinals (see for instance [1] [7]). These works seem to suggest that the limits of predicative mathematics could be way higher than what Feferman had in mind. It might be interesting to explore if Feferman's foundational project, or at least the proof-theoretic strength of the theory of truth, can somehow be extended beyond Γ_0. In particular, the research question could be put in this way: are there axiomatic theories of truth based on KF that can justify mathematical theories of proof-theoretic strength above Γ_0?

One way to achieve this is by iterating the theories of truth, as in [5]. I argue that this strategy, although logically sophisticated and of great proof-theoretic interest, goes against Feferman's foundational programme, which crucially relies on the notion of truth to dispense of transfinite iterations.

We survey the literature and look at two proposals that go in this direction. First, in [3], we can find an extension of KF with a principle of Generalized Induction, KFGI. Cantini points towards a proof-theoretic analysis of the theory but omits the details, in particular, no proof of the upper bound is presented. Therefore, we first provide a lower bound by following Cantini's suggestion, i.e. by embedding the theory ID_1 into the theory of truth. This is done via a translation that preserves the arithmetical theorems and interprets set-theoretic membership as truth predication. For the upper bound we provide the proof missing from Cantini's paper with an original strategy, i.e. by applying methods from impredicative proof theory as in [6]. We embed the theory KFGI in a semi-formal system with suitable rules. Via a collapsing lemma, we obtain cut-free proofs for the system with height $< \Psi(\varepsilon_{\Omega+1})$. By formalising this in a suitable system, combined with the lower bound, we prove that the ordinal of the theory is the Bachmann-Howard ordinal.

As a second example, we study the theory KF_μ, presented in [2]. The theory can be seen as an axiomatization of the minimal fixed point of Kripke's semantic construction. To achieve this, Burgess adds to the theory a minimality schema which states that the truth predicate of KF is the smallest among the truth predicates closed under its axioms. First, we observe that a proof-theoretic upper bound can be obtained via an embedding of the theory of truth in ID_1. This is achieved by translating the truth predicate as the set-constant corresponding to the minimal fixed-point of the Kripkean semantics positive operator based on strong-Kleene logic. For the lower-bound, we depart from Burgess' strategy and provide proof of GI into the theory, crucially employing the minimality schema. This is done to also establish a relationship between the two variants of KF we studied.

The two theories studied are extensions of KF that justify greater portions of mathematics compared to the original theory. In particular, they reach the ordinal of the theory ID_1, which is sometimes labelled as the first impredicative theory. This overcomes the proof-theoretic limits of the versions of KF introduced by Feferman. To conclude, it is important to remark that Feferman's project is philosophically solid: it relies on an understanding of Gödel incompleteness results and the literature on iterating reflection principles is vast and philosophically rich. On the other hand, whether the principles we surveyed have a comparatively solid philosophical justification constitutes an open problem.

References.

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