Paraconsistent arithmetic and recapture

Proponents of paraconsistent and relevant logics have long being focusing their attention to the development of systems of arithmetic which overcome some of the purported shortcomings of classical reasoning. However, arithmetic and its consequences seem to be entangled with classical logic, such that any non-classical theorist must seriously consider the question of how much proof-theoretic strength is lost by adopting a non-classical arithmetic.

Famously, Friedman and Meyer (1992) show that the relevant system $\mathtt{R}^{\#}$ is properly weaker than classical PA, while the system $\mathtt{R}^{\#\#}$, obtain by adding an omega-rule to $\mathtt{R}^{\#}$, proves all the theorems of classical Peano Arithmetic (PA) *in the classical, i.e. arrow free, language.* Nevertheless, this result has to be qualified: since the meaning of connectives in the arrow-free vocabulary of $\mathtt{R}^{\#\#}$ differs from the meaning of the respective connectives in classical logic, it is unclear whether the claim that theorems of PA are proved in the arrow-free fragment of the language of $\mathtt{R}^{\#\#}$ fully counts as recovering the strength of classical arithmetic. Then, a thorough assessment of whether non-classical arithmetics can recover classical strength has to be based on proof-theoretic analysis.[1] This is rendered especially difficult in paraconsistent systems because of the necessity to construct a system of ordinal notations in a paraconsistent setting.

I carry out this endeavour using Zach Weber's paraconsistent arithmetic based on the logic `subDLQ`, which presents a strong conditional and, contrarily to other systems of paraconsistent arithmetic, has no finite models. I adapt a construction of ordinal notations to the inconsistent setting using techniques anticipated by Weber (2016). Then, I show that, while the logic supports the proof of transfinite induction up to any ordinal less than ϵ_0 for arithmetical and non-arithmetical (possibly paraconsistent) predicates in the theory, because of the functioning of ordinal notations in the paraconsistent setting, the proof-theoretic strength of PA is recovered only if identity is assumed to behave classically. Otherwise, the paraconsistent system is shown to be strictly weaker than the classical one.

These results are discussed in the context of the recent debate about recapture theorems (see Fiore and Rosenblatt 2023, Nicolai 2022). Recapture theorems claim that, when reasoning with theories which reflect on arithmetic, such as axiomatic theories of truth, the purely arithmetical consequences of the theory should behave classically, because for the arithmetical language the meaning of connectives is assumed to collapse to the classical one, thus losing no proof-theoretic power. Nevertheless, I show that however crucial to the recovery of proof-theoretic strength for non-classical theories, this is an unwarranted assumption.

[1] For a non-paraconsistent logic, this has already been done by Fischer et al. (2023).