Topological semantics of the predicate modal calculus QGL extended with non-well-founded proofs *

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As shown by Montagna [1], the first-order predicate version of the Gödel-Löb modal logic GL denoted by QGL is neither arithmetically complete nor complete with respect to its Kripke semantics. In the present talk, we discuss an extension of QGL obtained by allowing non-well-derivations and consider topological semantics of the given calculus.

A *non-well-founded derivation, or* ∞*-derivation,* is a (possibly infinite) tree whose nodes are marked by predicate modal formulas and that is constructed according to the rules (mp), (gen) and (nec). In addition, any infinite branch in an ∞-derivation must contain infinitely many applications of the rule (nec). Below is an example of an ∞-derivation:

$$\text{mp}\;\dfrac{\text{nec}\;\dfrac{\text{gen}\;\dfrac{\text{mp}\;\dfrac{\begin{matrix}\vdots\\ \Box\forall x_2\, P_2(x_2)\end{matrix} \qquad \Box\forall x_2\, P_2(x_2) \to P_1(x_1)}{P_1(x_1)}}{\forall x_1\, P_1(x_1)}}{\Box\forall x_1\, P_1(x_1)} \qquad \Box\forall x_1\, P_1(x_1) \to P_0(x_0)}{P_0(x_0)}\;.$$

A *non-well-founded proof, or* ∞*-proof*, is an ∞-derivation, where all leaves are marked by axioms of QGL. We write $\mathsf{QGL}_\infty \vdash A$ if there is an ∞-proof with the root marked by A.

Our main result is that QGL_∞ is complete with respect to the corresponding class of predicate topological frames with constant domains (while QGL is not). A *predicate topological frame for* QGL_∞ is a tuple (X, τ, D), where (X, τ) is a scattered topological space and D is a non-empty domain. In contrast to the case of relational frames, the constant domain condition does not imply validity of the Barcan formula in the given semantics.

We prove weak topological completeness of QGL_∞ by means of a proof-theoretic presentation of the system in a form of a sequent calculus allowing non–well-founded proofs. Our proof is inspired by two sequent-based completeness arguments: the argument for classical predicate logic based on reduction trees and the argument for the system GL extended with non-well-founded derivations [2]. We also prove strong local topological completeness of QGL_∞ by generalizing Shehtman's ultrabouquet construction from [3].

Finally, it remains to stress that this is a joint work in progress with Pavel Razumnyy.

References

[1] F. Montagne, *The predicate modal logic of provability*, Notre Dame Journal of Formal Logic **25** (1987), 179–189.

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[3] V. Shehtman, *On neighbourhood semantics thirty years later*, We Will Show Them! Essays in Honour of Dov Gabbay (S. Artemov et al., ed.), vol. 2, College Publications, London, 2005, pp. 663–692.