A classification of the set-theoretic total recursive functions of KPI

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KP and KPI

Axioms of KP in the language $\mathcal{L} = \{\in, \notin\}$:

1. Logical Axioms: $\Gamma, A, \neg A$ for any formula A,
2. Leibniz Principle: $\Gamma, a = b \wedge B(a) \to B(b)$ for any formula B,
3. Pair: $\Gamma, \exists z(a \in z \wedge b \in z)$,
4. Union: $\Gamma, \exists z \forall y \in a \forall x \in y(x \in z)$,
5. Δ_0-Separation:

   $$\Gamma, \exists y[\forall x \in y(x \in a \wedge B(x)) \wedge \forall x \in a(B(x) \to x \in y)],$$

   for any Δ_0-formula B.
6. Class Induction: $\Gamma, \forall x[\forall y \in x B(y) \to B(x)] \to \forall x\ B(x)$ for any formula B,
7. Infinity: $\Gamma, \exists x[\exists z \in x(z \in x) \wedge \forall y \in x \exists z \in x(y \in z)]$,
8. Δ_0-Collection: $\Gamma, \forall x \in a \exists y B(x, y) \to \exists z \forall x \in a \exists y \in z\ B(x, y)$ for any Δ_0-formula B.

Axioms of KPI in the language $\mathcal{L}' = \{\in, \notin, Ad, \neg Ad\}$:

1. Logical Axioms,
2. Leibniz Principle,
3. Pair,
4. Union,
5. Δ_0-Separation,
6. Class Induction,
7. Infinity,
8. Ad1: $\Gamma, \forall x[Ad(x) \to \omega \in x \wedge \ Tran(x)]$,
9. Ad2: $\Gamma, \forall x \forall y[Ad(x) \wedge Ad(y) \to x \in y \vee x = y \vee y \in x]$,
10. Ad3: $\Gamma, \forall x[Ad(x) \to (Pair)^x \wedge (Union)^x \wedge (\Delta_0 - Sep)^x \wedge (\Delta_0 - Coll)^x]$,
11. Lim: $\Gamma, \forall x \exists y[Ad(y) \wedge x \in y]$.

The rules of inference are the following.

$$(\wedge)\ \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}$$

$$(\vee)\ \frac{\Gamma, A}{\Gamma, A \vee B} \qquad\qquad (\vee)\ \frac{\Gamma, B}{\Gamma, A \vee B}$$

$$(b\exists)\ \frac{\Gamma, a \in b \wedge B(a)}{\Gamma, \exists x \in b\ B(x)} \qquad\qquad (\exists)\ \frac{\Gamma, B(a)}{\Gamma, \exists x\ B(x)}$$

$$(b\forall)\ \frac{\Gamma, a \in b \to B(a)}{\Gamma, \forall x \in b\ B(x)} \qquad\qquad (\forall)\ \frac{\Gamma, B(a)}{\Gamma, \forall x\ B(x)}$$

$$(Cut)\ \frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

The Main Theorem

Definition

Let X be any set. We define for every ordinal α the set $L_\alpha(X)$ as:

$L_0(X) = TC(\{X\})$,
$L_{\alpha+1}(X) = \{Y \subseteq L_\alpha(X) : Y$ is definable over $\langle L_\alpha(X), \in\rangle\}$,
$L_\gamma(X) = \bigcup_{\alpha<\gamma} L_\alpha(X)$ if γ is a limit.

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Theorem (Main Theorem)

Let f be a set-recursive function such that KPI *proves that f is total and uniformly Σ-definable in any admissible set. Then, there is some natural number n such that*

$$V \models \forall x (f(x) \in \hat{G}_n(x)).$$

The premises of the theorem say:

$$\mathrm{KPI} \vdash Ad(u) \rightarrow [\forall x \in u \exists! y \in u \; A_f(x, y)^u].$$

The Ordinal Notation System

The proof of the main theorem relies on the (relativized) *ordinal analysis* of KPI. The ordinal analysis of a theory T assigns to T the ordinal α, the proof-theoretic ordinal of T: α is the supremum of the ordinals β such that T proves transfinite induction up to β.

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The set X is a fixed set. The set-theoretic rank of X is θ. The sequence $\langle \Omega_n : n \leq \omega \rangle$ enumerates the first "$\omega + 1$-many" uncountable regular cardinals κ such that $\kappa > \theta$.

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$$0, (1, 2, \ldots, \omega, \omega + 1, \ldots), \Gamma_0, \Gamma_1, \Gamma_2, \ldots, \Gamma_\theta, (\Gamma_{\theta+1}, \ldots), \Omega_0, \Omega_1, \ldots, \Omega_\omega.$$

For each β, we have $\delta, \zeta < \Gamma_\beta \to \varphi_\delta(\zeta) < \Gamma_\beta$.

Definition

For every α, for every $n < \omega$, we define $B_n(\alpha)$ by induction on n.

- $B_0(\alpha)$ is the closure of $\{0\} \cup \{\Gamma_\beta : \beta \leq \theta\} \cup \{\Omega_m : m \leq \omega\}$ under $+, \varphi.(\cdot)$ and $\psi_k \restriction \alpha$ for every $k < \omega$.

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- $B_{n+1}(\alpha)$ is the closure of $\Omega_n \cup \{\Omega_m : m \leq \omega\}$ under $+, \varphi.(\cdot)$ and $\psi_k \restriction \alpha$ for every $k < \omega$.

The ordinal collapsing function ψ_n is defined as $\psi_n(\alpha) = min\{\beta : \beta \notin B_n(\alpha)\}$.

Lemma

For every ordinal α *and every natural number n, we have:*

1. $\psi_n(\alpha)$ *is a strongly critical ordinal,*
2. $\Gamma_{\theta+1} \leq \psi_0(\alpha) < \Omega_0$ *and* $\Omega_n < \psi_{n+1}(\alpha) < \Omega_{n+1}$.

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A picture of $B_n(\alpha)$:

$$0, 1, \ldots, \omega, \omega + 1, \ldots, \Omega_n, \Omega_n + 1, \ldots, \psi_{n+1}(\alpha), \psi_{n+1}(\alpha) + 1, \ldots,$$
$$\Omega_{n+1}, \ldots, \ldots, \Omega_{n+2}, \ldots, \ldots, \Omega_\omega, \ldots, \ldots$$

Definition

Let α be an ordinal. We define the normal form of α as follows.

1. $\alpha =_{NF} \alpha_1 + \cdots + \alpha_n$ iff $\alpha = \alpha_1 + \cdots + \alpha_n$, $n > 1$, where the ordinals $\alpha_1, \ldots, \alpha_n$ are written in normal form and are additive principal and $\alpha_1 \geq \cdots \geq \alpha_n$,
2. $\alpha =_{NF} \varphi\alpha_1\alpha_2$ iff $\alpha = \varphi\alpha_1\alpha_2$ with $\alpha_1, \alpha_2 < \alpha$ and α_1, α_2 are written in normal form,
3. $\alpha =_{NF} \psi_n(\alpha_1)$ iff $\alpha = \psi_n(\alpha_1)$ with $\alpha_1 \in B_n(\alpha_1)$ and α_1 is written in normal form.

Definition

We define $T(\theta)$ as the set of strings in the language $\{0, +, \varphi\} \cup \{\Gamma_\beta : \beta < \theta\} \cup \{\Omega_n : n \leq \omega\} \cup \{\psi_n : n < \omega\}$ corresponding to ordinals written in normal form from the closure of $\{0\} \cup \{\Gamma_\beta : \beta < \theta\} \cup \{\Omega_n : n \leq \omega\}$ under $+, \varphi, \psi_n$ for $n < \omega$.

Theorem

The set $T(\theta)$ and the order $\prec$ on $T(\theta)$ induced by the ordering of ordinals are primitive recursive in θ.

From now on, we consider that all the ordinals belong to $T(\theta)$.

The system $\mathrm{RS}_l(X)$

Definition

The set $\mathcal{T}$ of $\mathrm{RS}_l(X)$-terms is defined as follows. Each term has an ordinal level.

- $\overline{u} \in \mathcal{T}$ for every $u \in TC(\{X\})$ and $|\overline{u}| = \Gamma_{\mathrm{rank}(u)}$.
- $\mathbb{L}_\alpha(X) \in \mathcal{T}$ for every $\alpha \leq \Omega_\omega$ and $|\mathbb{L}_\alpha(X)| = \Gamma_{\theta+1} + \alpha$.
- $[x \in \mathbb{L}_\alpha(X) : B(x, s_1, \ldots, s_n)^{\mathbb{L}_\alpha(X)}] \in \mathcal{T}$ for every $\alpha < \Omega_\omega$, for every KPl-formula $B(x, y_1, \ldots, y_n)$ and every $s_1, \ldots, s_n \in \mathcal{T}$ with $|s_1|, \ldots, |s_n| < \Gamma_{\theta+1} + \alpha$. Moreover, $|[x \in \mathbb{L}_\alpha(X) : B(x, s_1, \ldots, s_n)^{\mathbb{L}_\alpha(X)}]| = \Gamma_{\theta+1} + \alpha$.

In particular, we have $|\mathbb{L}_{\Omega_n}(X)| = \Omega_n$ for every $n \leq \Omega_\omega$.

Definition

The $RS_l(X)$-formulas are exactly the KPI-formulas replacing free variables by $RS_l(X)$-terms and restricting all unbounded quantifiers to $RS_l(X)$-terms.
The $RS_l(X)$-formulas of the form $\overline{u} \in \overline{v}$ or $\overline{u} \notin \overline{v}$ are called basic.

We will say that a formula $A(s_1, \ldots, s_n)^{\mathbb{L}_{\Omega_n}(X)}$ is Σ^{Ω_n} iff $A(x_1, \ldots, x_n)$ is a KPI Σ-formula and $|s_1|, \ldots, |s_n| < \Omega_n$.

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We will say that a formula $A(s_1, \ldots, s_n)^{\mathbb{L}_{\Omega_n}(X)}$ is Σ^{Ω_n} iff $A(x_1, \ldots, x_n)$ is a KPl Σ-formula and $|s_1|, \ldots, |s_n| < \Omega_n$.

For example, from the KPl-formula $\forall x{\in}y\, \exists z\, (x{\in}z)$ we get the Σ^{Ω_1}-formula $\forall x{\in}\, \mathbb{L}_{\Omega_0}(X)\, \exists z{\in}\mathbb{L}_{\Omega_1}(X)\, (x{\in}z)$.

We will use the following abbreviations.

Definition

1. $s = t$ will stand for $\forall x \in s(x \in t) \wedge \forall x \in t(x \in s)$.
2. $\neg A$ is obtained from A by replacing $\in$ by $\notin$ and vice-versa, $\vee$ by $\wedge$ and vice-versa, $\forall$ by $\exists$ and vice-versa and $Ad(\cdot)$ by $\neg Ad(\cdot)$ and vice-versa.
3. $A \rightarrow B$ will stand for $\neg A \vee B$.
4. Let s and t be terms such that $|s| < |t|$. For $\circ \in \{\wedge, \rightarrow\}$, we define

$$s \mathrel{\dot{\in}} t \circ A(s,t) = \begin{cases} \overline{u} \in \overline{v} \circ A(\overline{u}, \overline{v}) & \text{if } s \in t \equiv \overline{u} \in \overline{v}, \\ A(s,t) & \text{if } t = \mathbb{L}_\alpha(X), \\ B(s) \circ A(s,t) & \text{if } t = [x \in \mathbb{L}_\alpha(X) : B(x)]. \end{cases}$$

Definition

An operator is a function $\mathcal{H} : \mathcal{P}(ON) \to \mathcal{P}(ON)$ such that for every $Y, Y' \in \mathcal{P}(ON)$ the following conditions are satisfied.

1. $\{0\} \cup \{\Gamma_\beta : \beta \leq \theta + 1\} \cup \{\Omega_i : i \leq \omega\} \subseteq \mathcal{H}(Y)$.
2. Let $\alpha =_{NF} \alpha_1 + \cdots + \alpha_n$. Then, $\alpha \in \mathcal{H}(Y)$ iff $\alpha_1, \ldots, \alpha_n \in \mathcal{H}(Y)$.
3. Let $\alpha =_{NF} \varphi\alpha_1\alpha_2$. Then, $\alpha \in \mathcal{H}(Y)$ iff $\alpha_1, \alpha_2 \in \mathcal{H}(Y)$.
4. $Y \subseteq \mathcal{H}(Y)$.
5. If $Y \subseteq \mathcal{H}(Y')$ then $\mathcal{H}(Y) \subseteq \mathcal{H}(Y')$.

Moreover, $\mathcal{H}$ will often denote $\mathcal{H}(\emptyset)$.

Let $\mathcal{H}$ be an operator and let Γ be a set of formulas. We have that Γ is derived by an $\mathcal{H}$-controlled derivation with ordinal α whenever $\{\alpha\} \cup k(\Gamma) \subseteq \mathcal{H}$ and one of the following axioms or rules can be applied.

Axioms:

$$\mathcal{H} \overset{\alpha}{\vdash} \Gamma, \overline{u} \in \overline{v} \text{ for any } u, v \in TC(\{X\}) \text{ such that } u \in v,$$

$$\mathcal{H} \overset{\alpha}{\vdash} \Gamma, \overline{u} \notin \overline{v} \text{ for any } u, v \in TC(\{X\}) \text{ such that } u \notin v.$$

Rules:

$$(\wedge)\ \frac{\mathcal{H}\left|\frac{\alpha_0}{}\ \Gamma, A \wedge B, A \qquad \mathcal{H}\right|\frac{\alpha_1}{}\ \Gamma, A \wedge B, B}{\mathcal{H}\left|\frac{\alpha}{}\ \Gamma, A \wedge B\right.} \qquad \alpha_0, \alpha_1 < \alpha$$

$$(\vee)\ \frac{\mathcal{H}\left|\frac{\alpha_0}{}\ \Gamma, A \vee B, A\right.}{\mathcal{H}\left|\frac{\alpha}{}\ \Gamma, A \vee B\right.} \qquad \alpha_0 < \alpha$$

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$$(\in)\ \frac{\mathcal{H} \overset{\alpha_0}{\vdash} \Gamma, r \in t, s \,\dot{\in}\, t \wedge r = s}{\mathcal{H} \overset{\alpha}{\vdash} \Gamma, r \in t}$$

$\alpha_0 < \alpha$,
$|s| < |t|$,
$|s| < \Gamma_{\theta+1} + \alpha$,
$r \in t$ not basic.

$$(\notin)\ \frac{\mathcal{H}[s] \overset{\alpha_s}{\vdash} \Gamma, r \notin t, s \,\dot{\in}\, t \to r \neq s \text{ for all } |s| < |t|}{\mathcal{H} \overset{\alpha}{\vdash} \Gamma, r \notin t}$$

$\alpha_s < \alpha$,
$r \in t$ not basic.

$$(b\exists)\ \frac{\mathcal{H} \overset{\alpha_0}{\vdash} \Gamma, \exists x \in t\ B(x), s \,\dot{\in}\, t \wedge B(s)}{\mathcal{H} \overset{\alpha}{\vdash} \Gamma, \exists x \in t\ B(x)} \qquad \begin{array}{l} \alpha_0 < \alpha, \\ |s| < |t|, \\ |s| < \Gamma_{\theta+1} + \alpha. \end{array}$$

$$(b\forall)\ \frac{\mathcal{H}[s] \overset{\alpha_s}{\vdash} \Gamma, \forall x \in t\ B(x), s \,\dot{\in}\, t \to B(s) \text{ for all } |s| < |t|}{\mathcal{H} \overset{\alpha}{\vdash} \Gamma, \forall x \in t\ B(x)} \qquad \alpha_s < \alpha$$

$$(Ad)\ \frac{\mathcal{H} \overset{\alpha_0}{\vdash} \Gamma, Ad(t), t = \mathbb{L}_{\Omega_n}(X)}{\mathcal{H} \overset{\alpha}{\vdash} \Gamma, Ad(t)} \qquad \begin{array}{l} \alpha_0 < \alpha, \\ n \leq \omega, \\ \Omega_n < |t|. \end{array}$$

$$(\neg Ad)\ \frac{\mathcal{H} \overset{\alpha_n}{\vdash} \Gamma, \neg Ad(t), t \neq \mathbb{L}_{\Omega_n}(X) \text{ for all } n \leq \omega}{\mathcal{H} \overset{\alpha}{\vdash} \Gamma, \neg Ad(t)} \qquad \alpha_n < \alpha$$

$$(Cut)\ \frac{\mathcal{H} \overset{\alpha_0}{\vdash} \Gamma, A \qquad \mathcal{H} \overset{\alpha_0}{\vdash} \Gamma, \neg A}{\mathcal{H} \overset{\alpha}{\vdash} \Gamma} \qquad \alpha_0 < \alpha$$

$$(\mathrm{Ref}_n)\ \frac{\mathcal{H} \overset{\alpha_0}{\vdash} \Gamma, \exists z \in \mathbb{L}_{\Omega_n}(X)\ A^z, A^{\mathbb{L}_{\Omega_n}(X)}}{\mathcal{H} \overset{\alpha}{\vdash} \Gamma, \exists z \in \mathbb{L}_{\Omega_n}(X)\ A^z}$$

$\alpha_0, \Omega_n < \alpha$,
A is a Σ formula.

Definition

We define the rank of a term or formula by recursion.

- $\mathrm{rk}(\overline{u}) = \Gamma_{\mathrm{rank}(u)}$,
- $\mathrm{rk}(\mathbb{L}_\alpha(X)) = \Gamma_{\theta+1} + \omega \cdot \alpha$,
- $\mathrm{rk}([x \in \mathbb{L}_\alpha(X) : B(x)]) = \max(\Gamma_{\theta+1} + \omega \cdot \alpha + 1, \mathrm{rk}(B(\overline{\emptyset})) + 2)$,

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- $\mathrm{rk}([x \in \mathbb{L}_\alpha(X) : B(x)]) = \max(\Gamma_{\theta+1} + \omega \cdot \alpha + 1, \mathrm{rk}(B(\overline{\emptyset})) + 2)$,
- $\mathrm{rk}(s \in t) = \mathrm{rk}(s \notin t) = \max(\mathrm{rk}(s) + 6, \mathrm{rk}(t) + 1)$,
- $\mathrm{rk}(Ad(t)) = \mathrm{rk}(\neg Ad(t)) = \mathrm{rk}(t) + 5$,
- $\mathrm{rk}(A \vee B) = \mathrm{rk}(A \wedge B) = \max(\mathrm{rk}(A), \mathrm{rk}(B)) + 1$,
- $\mathrm{rk}(\exists x \in t\ A(x)) = \mathrm{rk}(\forall x \in t\ A(x)) = \max(\mathrm{rk}(t), \mathrm{rk}(A(\overline{\emptyset})) + 2)$.

Lemma

Let A be a formula. Then $\mathrm{rk}(B) < \mathrm{rk}(A)$ for any premise B of A.

Definition

We will write $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma$ whenever $\mathcal{H} \overset{\alpha}{\vdash} \Gamma$ and all the cut formulas in the proof have rank strictly less than ρ.

Lemma (Predicative Cut Elimination)

Let $\alpha \in \mathcal{H}$. Let ρ be an ordinal such that $\Omega_n \notin [\rho, \rho + \omega^\alpha)$ for any $n < \omega$. If $\mathcal{H} \overset{\beta}{\underset{\rho+\omega^\alpha}{\vdash}} \Gamma$ then $\mathcal{H} \overset{\varphi\alpha\beta}{\underset{\rho}{\vdash}} \Gamma$.

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We will write $\mathcal{H} \vdash^{\alpha}_{\rho} \Gamma$ whenever $\mathcal{H} \vdash^{\alpha} \Gamma$ and all the cut formulas in the proof have rank strictly less than ρ.

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Example 1. Let $\mathcal{H} \vdash^{\beta}_{\Omega_n+\omega^{\alpha}} \Gamma$ with $\Omega_n < \Omega_n + \omega^{\alpha} < \Omega_{n+1}$. Then, we get $\mathcal{H} \vdash^{\varphi\alpha\beta}_{\Omega_n+1} \Gamma$.

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We will write $\mathcal{H} \left|\frac{\alpha}{\rho}\right. \Gamma$ whenever $\mathcal{H} \left|\frac{\alpha}{}\right. \Gamma$ and all the cut formulas in the proof have rank strictly less than ρ.

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Example 2. Let $\mathcal{H} \left|\frac{\beta}{\omega^\alpha}\right. \Gamma$ with $\alpha < \Omega_0$. Then, we get $\mathcal{H} \left|\frac{\varphi\alpha\beta}{0}\right. \Gamma$.

Definition

For any set of ordinal Y we let

$$\mathcal{H}_\beta(Y) = \bigcap\{B_n(\alpha) : Y \subseteq B_n(\alpha) \text{ with } \beta < \alpha \text{ and } n < \omega\}.$$

We recall that $A^{\mathbb{L}_{\Omega_m}(X)}$ is Σ^{Ω_m} iff A is a KPI Σ-formula and the terms replacing free variables have level less than Ω_m.

Theorem (Collapsing Theorem)

Let $n \leq \omega$ and let $m < \omega$. Let Γ be a set of Σ^{Ω_m}-formulas and let α and β be ordinals with $\beta \in \mathcal{H}_\beta$.

If $\mathcal{H}_\beta \left|\frac{\alpha}{\Omega_n+1}\right. \Gamma$ then $\mathcal{H}_{\beta+\omega^{\Omega_n+1+\alpha}} \left|\frac{\psi_m(\beta+\omega^{\Omega_n+1+\alpha})}{\psi_m(\beta+\omega^{\Omega_n+1+\alpha})}\right. \Gamma$.

The Embedding Theorem

Lemma

Let $\mathcal{H}$ be any operator.
For every axiom Ax *of* KPI, *we have* $\mathcal{H} \frac{\alpha}{\rho} (\mathrm{Ax})^{\mathbb{L}_{\Omega_\omega}(X)}$ *where* $\rho \leq \Omega_\omega$ *and* $\alpha \leq \Omega_\omega \cdot \omega^2$.

Each rule of KPI can be embedded in $\mathrm{RS}_I(X)$. The application of an embedded KPI rule increases the cut-complexity of the derivation by a finite number.

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Each rule of KPl can be embedded in $\mathrm{RS}_l(X)$. The application of an embedded KPl rule increases the cut-complexity of the derivation by a finite number.

Theorem (Embedding)

Let $\Gamma(a_1, \ldots, a_n)$ *be a finite set of formulas with all the free variables displayed such that* KPl $\vdash \Gamma(a_1, \ldots, a_n)$. *Then, there is some* $m < \omega$ *such that for any operator* $\mathcal{H}$ *and any terms* $s_1, \ldots, s_n$ *controlled by* $\mathcal{H}$ *we have*

$$\mathcal{H} \left|\frac{\Omega_\omega \cdot \omega^m}{\Omega_\omega + m}\right. \Gamma(s_1, \ldots, s_n)^{\mathbb{L}_{\Omega_\omega}(X)}.$$

The proof of the Main Theorem

Definition

We define the ordinal e_n by recursion on n as follows:

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Now, we define for each $n < \omega$ the set $\hat{G}_n(x) = L_{\psi_0(e_{n+3})}(x)$.

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Lemma

For every natural number n we have $e_n \in B_0(e_{n+1})$.

Theorem (Main Theorem)

Let f *be a set-recursive function such that* KPI *proves that* f *is total and uniformly* Σ*-definable in any admissible set. Then, there is some* $n < \omega$ *such that*

$$V \vDash \forall x(f(x) \in \hat{G}_n(x)).$$

Proof.

KPI $\vdash Ad(u) \rightarrow [\forall x \in u \exists! y \in u \ A_f(x, y)^u]$. Fix X and let θ be the set-theoretic rank of X.

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$$V \models \forall x(f(x) \in \hat{G}_n(x)).$$

Proof.

KPI $\vdash Ad(u) \to [\forall x \in u \exists! y \in u\ A_f(x,y)^u]$. Fix X and let θ be the set-theoretic rank of X. By the Embedding Theorem, we have
$\mathcal{H}_0 \left|\frac{\Omega_\omega \cdot \omega^m}{\Omega_\omega + m}\right. Ad(\mathbb{L}_{\Omega_0}(X)) \to \forall x \in \mathbb{L}_{\Omega_0}(X) \exists! y \in \mathbb{L}_{\Omega_0}(X) A_f(x,y)^{\mathbb{L}_{\Omega_0}(X)}$.

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Let f be a set-recursive function such that KPI *proves that f is total and uniformly* Σ*-definable in any admissible set. Then, there is some* $n < \omega$ *such that*

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By an application of the (Cut) rule, we get
$\mathcal{H}_0 \left|\frac{\Omega_\omega \cdot \omega^m}{\Omega_\omega + m}\right. \forall x \in \mathbb{L}_{\Omega_0}(X) \exists ! y \in \mathbb{L}_{\Omega_0}(X) A_f(x, y)^{\mathbb{L}_{\Omega_0}(X)}$.

By Inversion,

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By Predicative Cut Elimination,

$$\mathcal{H}_0 \left|\frac{e_{m+1}}{\Omega_\omega + 1}\right. \exists y \in \mathbb{L}_{\Omega_0}(X) A_f(X, y)^{\mathbb{L}_{\Omega_0}(X)}.$$

□

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By the Collapsing Theorem,

$$\mathcal{H}_{e_{m+2}} \left|\frac{\psi_0(e_{m+2})}{\psi_0(e_{m+2})}\right. \exists y \in \mathbb{L}_{\Omega_0}(X) A_f(X, y)^{\mathbb{L}_{\Omega_0}(X)}.$$

□

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Let f be a set-recursive function such that KPI *proves that f is total and uniformly Σ-definable in any admissible set. Then, there is some $n < \omega$ such that*

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By an application of the (Cut) rule, we get
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By Predicative Cut Elimination, taking $\alpha = \varphi(\psi_0(e_{m+2}))(\psi_0(e_{m+2}))$,

$$\mathcal{H}_{e_{m+2}} \left|\frac{\alpha}{0}\right. \exists y \in \mathbb{L}_{\Omega_0}(X) A_f(X, y)^{\mathbb{L}_{\Omega_0}(X)}.$$

□

Theorem (Main Theorem)

Let f be a set-recursive function such that KPI *proves that f is total and uniformly Σ-definable in any admissible set. Then, there is some $n < \omega$ such that*

$$V \models \forall x(f(x) \in \hat{G}_n(x)).$$

Proof.

By an application of the (Cut) rule, we get
$\mathcal{H}_0 \left|\frac{\Omega_\omega \cdot \omega^m}{\Omega_\omega + m}\right. \forall x \in \mathbb{L}_{\Omega_0}(X) \exists! y \in \mathbb{L}_{\Omega_0}(X) A_f(x, y)^{\mathbb{L}_{\Omega_0}(X)}$.

By Boundedness,

$$\mathcal{H}_{e_{m+2}} \left|\frac{\alpha}{0}\right. \exists y \in \mathbb{L}_\alpha(X) A_f(X, y)^{\mathbb{L}_\alpha(X)}.$$

It follows that $L_\alpha(X) \models \exists y \; A_f(X, y)$. □

Theorem (Main Theorem)

Let f be a set-recursive function such that KPI *proves that f is total and uniformly Σ-definable in any admissible set. Then, there is some $n < \omega$ such that*

$$V \models \forall x(f(x) \in \hat{G}_n(x)).$$

Proof.

We have $f(X) \in L_\alpha(X)$, where $\alpha = \varphi(\psi_0(e_{m+2}))(\psi_0(e_{m+2}))$. But we have $e_{m+2} \in B_0(e_{m+3})$. Therefore, $\psi_0(e_{m+2}) < \psi_0(e_{m+3})$ and so $\alpha < \psi_0(e_{m+3})$.

Hence,

$$f(X) \in L_\alpha(X) \subseteq L_{\psi_0(e_{m+3})} = \hat{G}_{m+3}(X).$$

□

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