Two-dimensional Kripke Semantics

Alex Kavvos

University of Bristol

7th Workshop on Proof Theory, Modal Logic, and Reflection Principles (Wormshop 2024)
Ghent University, 3 September 2024

Kripke semantics vs. type theory

Modal logic is important in Computer Science:

- temporal logic
- epistemic logic
- dynamic logic
- Hennessy-Milner logic

In most cases, it is given a **Kripke semantics**.

But in type theory **proofs are important** (Curry-Howard-Lambek).

Type-theoretic **modalities** arise everywhere in programming:

- ‘logical’ time
- proof-irrelevance
- globality
- information flow

How can we connect these two worlds?

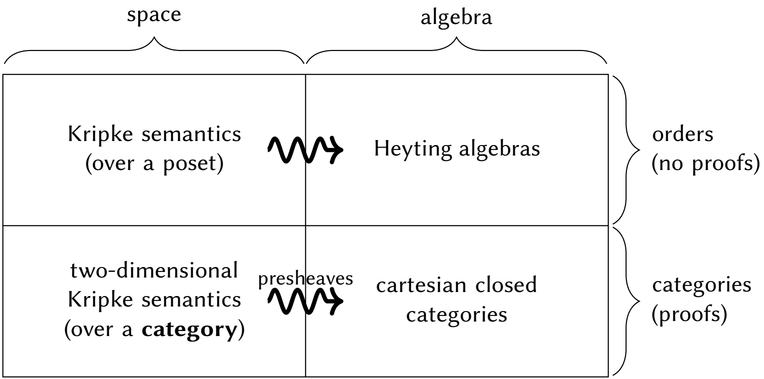
Models of intuitionistic logic

	space	algebra
orders (no proofs)	Kripke semantics (over a poset) ⇝	Heyting algebras
categories (proofs)		cartesian closed categories

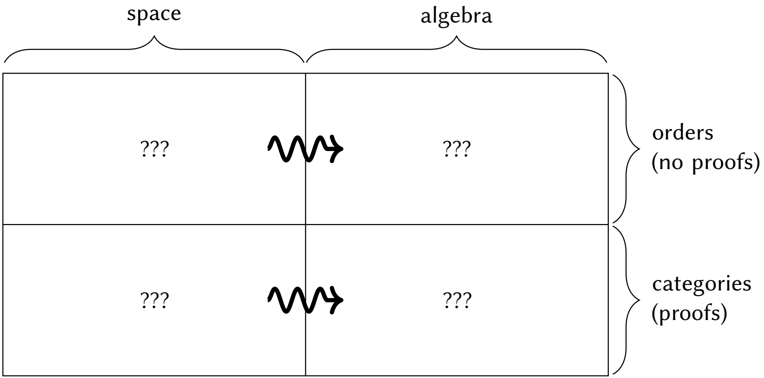
Models of intuitionistic logic

	space	algebra
orders (no proofs)	Kripke semantics (over a poset) $\rightsquigarrow$	Heyting algebras
categories (proofs)	two-dimensional Kripke semantics (over a **category**)	cartesian closed categories

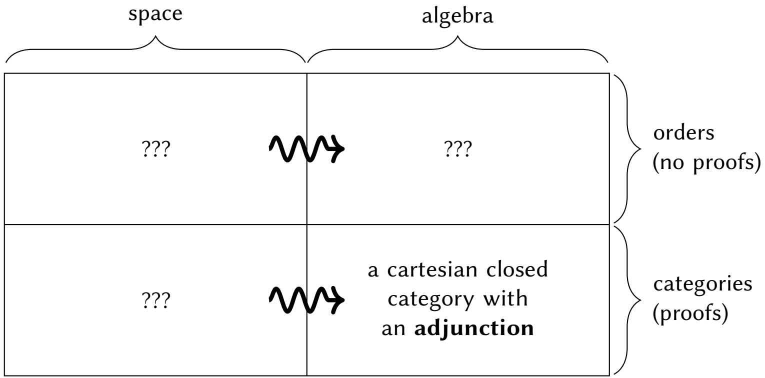
Models of intuitionistic logic



Models of **intuitionistic** modal logic



Models of **intuitionistic** modal logic



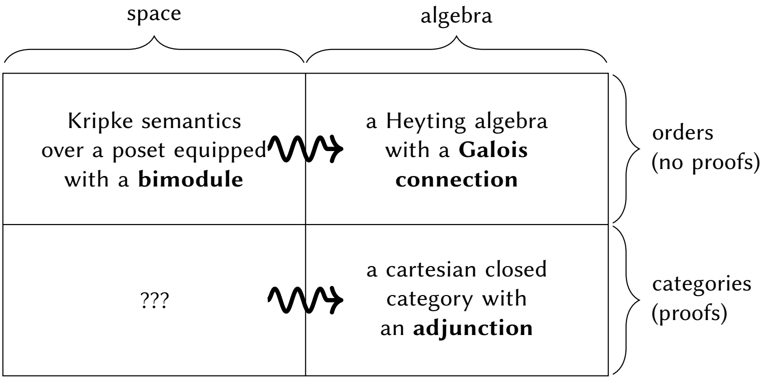
Using an adjunction was proposed by Clouston [Clo18].
It has proven remarkably robust in modal type theory.
This is an **objective** answer in the sense of Lawvere [Law94; LS09].

Models of **intuitionistic** modal logic

	space	algebra
orders (no proofs)	??? ⇝	a Heyting algebra with a **Galois connection**
categories (proofs)	??? ⇝	a cartesian closed category with an **adjunction**

Using an adjunction was proposed by Clouston [Clo18].
It has proven remarkably robust in modal type theory.
This is an **objective** answer in the sense of Lawvere [Law94; LS09].

Models of **intuitionistic** modal logic



Using an adjunction was proposed by Clouston [Clo18].
It has proven remarkably robust in modal type theory.
This is an **objective** answer in the sense of Lawvere [Law94; LS09].

Models of **intuitionistic** modal logic

	space		algebra
orders (no proofs)	Kripke semantics over a poset equipped with a **bimodule**	⇝	a Heyting algebra with a **Galois connection**
categories (proofs)	two-dimensional Kripke semantics over a category equipped with a **profunctor**	⇝	a cartesian closed category with an **adjunction**

Using an adjunction was proposed by Clouston [Clo18].
It has proven remarkably robust in modal type theory.
This is an **objective** answer in the sense of Lawvere [Law94; LS09].

Roadmap

Intuitionistic logic: Space vs. Algebra

Modal logic: Bimodules

Stable semantics

I. Intuitionistic logic: Space vs. Algebra

Kripke semantics of intuitionistic logic

Let $(W, \sqsubseteq)$ be a **Kripke frame**, i.e. a partial order.

$\mathsf{Up}(W)$ = **upper sets** $S \subseteq W$ (where $w \in S$ and $w \sqsubseteq v$ imply $v \in S$)

Let $V : \mathsf{Var} \to \mathsf{Up}(W)$ map each proposition to an upper set. Define

$$
\begin{aligned}
w \vDash p &\stackrel{\text{def}}{\equiv} w \in V(p) \\
w \vDash \bot &\stackrel{\text{def}}{\equiv} \text{never} \\
w \vDash \varphi \wedge \psi &\stackrel{\text{def}}{\equiv} w \vDash \varphi \text{ and } w \vDash \psi \\
w \vDash \varphi \vee \psi &\stackrel{\text{def}}{\equiv} w \vDash \varphi \text{ or } w \vDash \psi \\
w \vDash \varphi \to \psi &\stackrel{\text{def}}{\equiv} \forall v.\ w \sqsubseteq v \text{ and } v \vDash \varphi \text{ imply } v \vDash \psi
\end{aligned}
$$

Monotonicity: $w \vDash \varphi$ and $w \sqsubseteq v$ imply $v \vDash \varphi$

Theorem (Kripke)

A formula is valid (in all frames and all words) iff it is a theorem.

Algebraic semantics of intuitionistic logic

A **Heyting algebra** $(H, \leq)$ is a lattice (has finite meets and joins) such that for every $x, y \in H$ there exists $x \Rightarrow y \in H$ with

$$c \wedge x \leq y \quad \Longleftrightarrow \quad c \leq x \Rightarrow y \qquad \text{for all } c \in H$$

Suppose that for each proposition p we have an element $[\![p]\!] \in H$. Intuitionistic logic can then be interpreted into H compositionally:

$$\begin{aligned} [\![\bot]\!] &\stackrel{\text{def}}{=} 0 \\ [\![\varphi \wedge \psi]\!] &\stackrel{\text{def}}{=} [\![\varphi]\!] \wedge [\![\psi]\!] \\ [\![\varphi \vee \psi]\!] &\stackrel{\text{def}}{=} [\![\varphi]\!] \vee [\![\psi]\!] \\ [\![\varphi \to \psi]\!] &\stackrel{\text{def}}{=} [\![\varphi]\!] \Rightarrow [\![\psi]\!] \end{aligned}$$

Theorem

A formula is valid ($=1$ in all algebras) iff it is a theorem.

Prime algebraic lattices: from space to algebra

Let $(W, \sqsubseteq)$ be a **Kripke frame**, and $\mathbb{2} \stackrel{\text{def}}{=} \{0 \sqsubseteq 1\}$.

$[W, \mathbb{2}]$ (= monotone maps $W \to \mathbb{2}$) has many curious properties:

- $[W, \mathbb{2}] \cong \mathrm{Up}(W)$ where the order is inclusion
- It is a **complete Heyting algebra** (arbitrary joins and meets)
- The **principal upper set** embedding $\uparrow : W^{\mathrm{op}} \to [W, \mathbb{2}]$ given by $w \mapsto \{v \mid w \sqsubseteq v\}$ preserves meets and exponentials.
- An element is a **prime** ($p \sqsubseteq \bigsqcup_i d_i \Rightarrow \exists i.\ p \sqsubseteq d_i$) iff it is $\uparrow w$.
- Every upper set S is the join of primes below it:

$$S = \bigsqcup\{P \mid P \text{ prime}, P \subseteq S\} = \bigsqcup\{\uparrow w \mid w \in S\}$$

In short: $[W, \mathbb{2}]$ is a **prime algebraic lattice** [Win09].

A **duality** (Raney [Ran52]; Nielsen, Plotkin, and Winskel [NPW81]):

$$\mathbf{Pos}^{\mathrm{op}} \simeq \mathbf{PrAlgLatt}$$

Models of intuitionistic logic

	space		algebra
orders (no proofs)	Kripke semantics (over a poset)	⇝ upper sets ⇝	Heyting algebras
categories (proofs)	two-dimensional Kripke semantics (over a **category**)	⇝ presheaves ⇝	cartesian closed categories

Categories as spaces

A category $\mathcal{C}$ has

- **objects** $c, d, \ldots \in \mathcal{C}$
- **morphisms** $f, g, \ldots : c \to d$ between two objects
- a way to **compose** morphisms, and **identity** morphisms

Categories are often used as 'mathematical universes'
(sets, graphs, vector spaces, topological spaces, ...)

But a category can also be seen as a **partial order with evidence**.

$$\frac{}{x \sqsubseteq x} \qquad \frac{x \sqsubseteq y \qquad y \sqsubseteq z}{x \sqsubseteq z}$$

$$\frac{}{\mathrm{id}_x : x \to x} \qquad \frac{f : x \to y \qquad g : y \to z}{g \circ f : x \to z}$$

A category can also be seen as a *space* with *direction*.

Two-dimensional Kripke semantics of intuitionistic logic

Take any (small) category $\mathcal{C}$. Define a set

$$[\![\varphi]\!]_w$$

of **proofs** of φ at a world $w \in \mathcal{C}$, by induction on φ.

$$\begin{aligned}
[\![\bot]\!]_w &\stackrel{\text{def}}{=} \emptyset \\
[\![\varphi \wedge \psi]\!]_w &\stackrel{\text{def}}{=} [\![\varphi]\!]_w \times [\![\psi]\!]_w = \{(x, y) \mid x \in [\![\varphi]\!]_w, y \in [\![\psi]\!]_w\} \\
[\![\varphi \vee \psi]\!]_w &\stackrel{\text{def}}{=} [\![\varphi]\!]_w + [\![\psi]\!]_w = \{(1, a) \mid a \in [\![\varphi]\!]_w\} \cup \{(2, b) \mid b \in [\![\psi]\!]_w\} \\
[\![\varphi \to \psi]\!]_w &\stackrel{\text{def}}{=} (v : \mathcal{C}) \to \mathrm{Hom}_{\mathcal{C}}(w, v) \to [\![\varphi]\!]_v \to [\![\psi]\!]_v \quad \text{(not exactly)}
\end{aligned}$$

Monotonicity: for each $f : w \to v$ and $x \in [\![\varphi]\!]_w$ define $f \cdot x \in [\![\varphi]\!]_v$
This defines a **presheaf**, i.e. a functor

$$[\![\varphi]\!] : \mathcal{C} \longrightarrow \mathbf{Set}$$

Kripke semantics of intuitionistic logic

Let $(W, \sqsubseteq)$ be a **Kripke frame**, i.e. a partial order.

$\mathrm{Up}(W)$ = **upper sets** $S \subseteq W$ (where $w \in S$ and $w \sqsubseteq v$ imply $v \in S$)

Let $V : \mathrm{Var} \to \mathrm{Up}(W)$ map each proposition to an upper set. Define

$$
\begin{aligned}
w \vDash p &\overset{\text{def}}{\equiv} w \in V(p) \\
w \vDash \bot &\overset{\text{def}}{\equiv} \text{never} \\
w \vDash \varphi \land \psi &\overset{\text{def}}{\equiv} w \vDash \varphi \text{ and } w \vDash \psi \\
w \vDash \varphi \lor \psi &\overset{\text{def}}{\equiv} w \vDash \varphi \text{ or } w \vDash \psi \\
w \vDash \varphi \to \psi &\overset{\text{def}}{\equiv} \forall v.\ w \sqsubseteq v \text{ and } v \vDash \varphi \text{ imply } v \vDash \psi
\end{aligned}
$$

Monotonicity: $w \vDash \varphi$ and $w \sqsubseteq v$ imply $v \vDash \varphi$

Theorem (Kripke)

A formula is valid (in all frames and all words) iff it is a theorem.

Presheaves: from space to category

Play the same trick as before, but replace $\mathbb{2}$ by **Set** [Law73].

The category $[\mathcal{C}, \mathbf{Set}]$ of presheaves $\mathcal{C} \longrightarrow \mathbf{Set}$:

- is a **(co)complete cartesian closed category**
- The **Yoneda embedding** $\mathbf{y} : \mathcal{C}^{\mathrm{op}} \longrightarrow [\mathcal{C}, \mathbf{Set}]$ given by $\mathbf{y}(w) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(w, -)$ preserves products and exponentials.
- A presheaf P is **tiny** just if $\mathrm{Hom}(P, -)$ preserves colimits. All representables are tiny [and vice versa if $\mathcal{C}$ is Cauchy-complete].
- Every presheaf $P : \mathcal{C} \longrightarrow \mathbf{Set}$ is a colimit of tiny objects:

$$P = \varinjlim_{(w,x) \in \mathsf{el}\, P} \mathbf{y}(w)$$

There is a duality: $\mathbf{Cat}_{\mathrm{cc}}^{\mathrm{op}} \simeq \mathbf{PshCat}$ (Bunge's theorem).

Prime algebraic lattices: from space to algebra

Let $(W, \sqsubseteq)$ be a **Kripke frame**, and $\mathbb{2} \stackrel{\text{def}}{=} \{0 \sqsubseteq 1\}$.

$[W, \mathbb{2}]$ (= monotone maps $W \to \mathbb{2}$) has many curious properties:

- $[W, \mathbb{2}] \cong \mathrm{Up}(W)$ where the order is inclusion
- It is a **complete Heyting algebra** (arbitrary joins and meets)
- The **principal upper set** embedding $\uparrow : W^{\mathrm{op}} \to [W, \mathbb{2}]$ given by $w \mapsto \{v \mid w \sqsubseteq v\}$ preserves meets and exponentials.
- An element is a **prime** ($p \sqsubseteq \bigsqcup_i d_i \Rightarrow \exists i.\ p \sqsubseteq d_i$) iff it is $\uparrow w$.
- Every upper set S is the join of primes below it:

$$S = \bigsqcup \{P \mid P \text{ prime}, P \subseteq S\} = \bigsqcup \{\uparrow w \mid w \in S\}$$

In short: $[W, \mathbb{2}]$ is a **prime algebraic lattice** [Win09].

A **duality** (Raney [Ran52]; Nielsen, Plotkin, and Winskel [NPW81]):

$$\mathbf{Pos}^{\mathrm{op}} \simeq \mathbf{PrAlgLatt}$$

Duality

This construction gives us a duality

$$\mathbf{Cat}_{cc}^{op} \simeq \mathbf{PshCat}$$

between

- (Cauchy-complete, small) categories (≈ '2D Kripke frames')
- presheaf categories (≈ 'proof-relevant prime alg. lattices')

In short:

> A two-dimensional Kripke semantics is a categorical semantics in a presheaf category $[\mathcal{C}, \mathbf{Set}]$.

II. Modal logic: Bimodules

What is intuitionistic modal logic?

Not clear, in particular around $\Diamond$! (Das and Marin [DM23])

Consider accessibility relation $R \subseteq W \times W$ on the poset of worlds. How to make it compatible with $\sqsubseteq$?

The Simpson [Sim94] criteria:

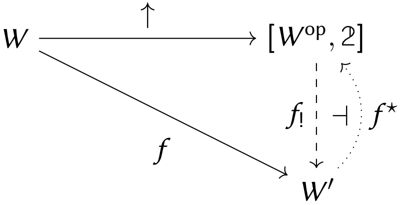
1. It should be conservative over intuitionistic logic.
2. It should prove all intuitionistic theorems (even with modalities).
3. Adding $\varphi \vee \neg\varphi$ should yield a classical modal logic.
4. It should satisfy the disjunction property.
5. $\Box$ and $\Diamond$ should be independent.
6. Its semantics should be 'intuitionistically comprehensible.'

#6 is formalised by translation to intuitionistic first-order logic.

An alternative proposal: **let category theory show you the way**.

Extensions

Let W' be a **complete lattice**, and let $f : W \to W'$ be monotone.



$f_!$: the **unique join-preserving** map satisfying $f_!(\uparrow w) = f(w)$.

$$f_!(S) \stackrel{\text{def}}{=} \bigsqcup \{f(w) \mid w \in S\}$$

As both lattices are complete, this has a right adjoint $f^\star$. Explicitly:

$$f^\star(w') \stackrel{\text{def}}{=} \{w \mid f(w) \sqsubseteq w'\}$$

Then

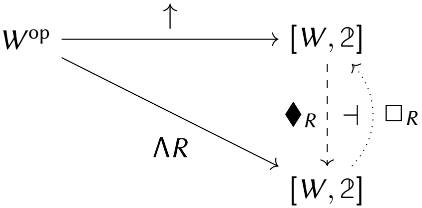
$$f_!(S) \sqsubseteq w' \quad \iff \quad S \subseteq f^\star(w')$$

Bimodules and Extensions

Let $(W, \sqsubseteq)$ be a Kripke frame. $R \subseteq W \times W$ is a **bimodule** just if

$$w' \sqsubseteq w \, R \, v \sqsubseteq v' \implies w' \, R \, v'$$

Equivalently: $R : W^{\mathrm{op}} \times W \to \mathbb{2}$. Now extend $\Lambda R : W^{\mathrm{op}} \to [W, \mathbb{2}]$:



Concretely: $\begin{cases} \blacklozenge_R(S) \stackrel{\text{def}}{=} \{w \in W \mid \exists v.\ v \, R \, w \text{ and } v \in S\} \\ \Box_R(S) \stackrel{\text{def}}{=} \{w \in W \mid \forall v.\ w \, R \, v \text{ implies } v \in S\} \end{cases}$

Every such adjunction on $[W, \mathbb{2}]$ corresponds to a bimodule!

Duality: $\mathbf{EBimod}^{\mathrm{op}} \simeq \mathbf{PrAlgLattO}$.

The logic of Dzik, Järvinen, and Kondo [DJK10]

A very simple **tense logic** with two modalities, $\blacklozenge$ and $\square$.

Kripke semantics:

$$w \models \blacklozenge\varphi \overset{\text{def}}{\equiv} \exists v.\ v\,R\,w \text{ and } v \models \varphi$$

$$w \models \square\varphi \overset{\text{def}}{\equiv} \forall v.\ w\,R\,v \text{ implies } v \models \varphi$$

Algebraic semantics: a Heyting algebra with a Galois connection.

$$\frac{\blacklozenge\varphi \to \psi}{\varphi \to \square\psi} \qquad \text{and} \qquad \frac{\varphi \to \square\psi}{\blacklozenge\varphi \to \psi}$$

Some derivable rules:

$$\frac{\varphi \to \psi}{\square\varphi \to \square\psi} \qquad \frac{\varphi}{\square\varphi} \qquad \frac{}{\square\top} \qquad \frac{\blacklozenge\bot}{\bot} \qquad \frac{\varphi \to \psi}{\blacklozenge\varphi \to \blacklozenge\psi}$$

The usual $\lozenge$ is **not monotonic** in this setting.

Lifting to categories

- Replace bimodules by **profunctors**
- Use **left Kan extension** along Yoneda

This leads to a duality $\mathbf{EProf}_{\mathrm{cc}}^{\mathrm{op}} \simeq \mathbf{PshCatO}$.

Modalities on presheaves $P : \mathcal{C} \longrightarrow \mathbf{Set}$:

$$(\blacklozenge P)(w) = \int^{v \in \mathcal{C}} R(v, w) \times P(v)$$

$$(\Box P)(w) = \int_{v \in \mathcal{C}} R(w, v) \to P(v)$$

Theorem

A two-dimensional Kripke semantics over $\mathcal{C}$ uniquely corresponds to

$$[\mathcal{C}, \mathbf{Set}] \underset{\blacklozenge}{\overset{\Box}{\underset{\top}{\rightleftarrows}}} [\mathcal{C}, \mathbf{Set}]$$

III. Stable semantics

Completeness?

The developments so far only prove **relative completeness**:

- Suppose a formula is valid in all Heyting algebras.
- Then it is valid in all prime algebraic lattices.
- Then it is valid in all Kripke semantics

$\therefore$ the algebraic semantics is as complete as the Kripke semantics.

How to get the opposite direction?

The classic proof (Gehrke and van Gool [Gv24, §4.4]):

- Make a Kripke frame of **prime filters** of the algebra.
- Show relative completeness with respect to that.

For this logic: Dzik, Järvinen, and Kondo [DJK10, §5].

But this is **non-constructive**, and also not very nice.

For a closer correspondence we have to 'tweak' Kripke semantics.

Stable semantics

Replace

- the poset of worlds by a **distributive lattice** $(W, \sqsubseteq)$
- upper sets by (non-prime) **filters**

$F \subseteq W$ is a **filter** just if it is an upper set and

$$1 \in F, \qquad x \in F \text{ and } y \in F \text{ imply } x \wedge y \in F$$

$$\begin{aligned}
w \vDash p &\overset{\text{def}}{\equiv} w \in V(p) \in \mathsf{Filt}(W) \\
w \vDash \bot &\overset{\text{def}}{\equiv} (1 \leq w) \qquad \text{(i.e. } w = 1\text{)} \\
w \vDash \varphi \wedge \psi &\overset{\text{def}}{\equiv} w \vDash \varphi \text{ and } w \vDash \psi \\
w \vDash \varphi \vee \psi &\overset{\text{def}}{\equiv} \exists v_1, v_2.\ v_1 \wedge v_2 \sqsubseteq w \text{ and } v_1 \vDash \varphi \text{ and } v_2 \vDash \psi \\
w \vDash \varphi \to \psi &\overset{\text{def}}{\equiv} \forall v.\ w \sqsubseteq v \text{ and } v \vDash \varphi \text{ imply } v \vDash \psi
\end{aligned}$$

This semantics is also sound and complete for intuitionistic logic!

Spectral locales: from space to algebra

Let $(W, \sqsubseteq)$ be a **distributive lattice**, and $2 \stackrel{\text{def}}{=} \{0 \sqsubseteq 1\}$.

$[W, 2]_\wedge$ (= $\wedge$-preserving $W \to 2$) has many curious properties:

- $[W, 2]_\wedge \cong \text{Filt}(W)$ where the order is inclusion
- It is a **complete Heyting algebra** (arbitrary joins and meets)
- The **principal filter** embedding $\uparrow : W^{\text{op}} \to [W, 2]_\wedge$ preserves finite meets, finite joins, and exponentials. Hence **any Heyting algebra H can be embedded in such a lattice**:

$$H \hookrightarrow [H^{\text{op}}, 2]_\wedge$$

- An elt. is **compact** ($p \sqsubseteq \bigsqcup^{\uparrow} X \Rightarrow \exists d \in X.\ p \sqsubseteq d$) iff it is $\uparrow w$.
- Every filter F is a directed supremum of compact ones:

$$F = \bigsqcup^{\uparrow} \{S \mid S \text{ compact}, S \subseteq F\} = \bigsqcup^{\uparrow} \{\uparrow w \mid w \in F\}$$

In short: $[W, 2]$ is a **spectral locale** (or a **coherent frame**)
(= algebraic cHA whose compact elts form a sub-lattice).

Prime algebraic lattices: from space to algebra

Let $(W, \sqsubseteq)$ be a **Kripke frame**, and $2 \stackrel{\text{def}}{=} \{0 \sqsubseteq 1\}$.

$[W, 2]$ (= monotone maps $W \to 2$) has many curious properties:

- $[W, 2] \cong \mathsf{Up}(W)$ where the order is inclusion
- It is a **complete Heyting algebra** (arbitrary joins and meets)
- The **principal upper set** embedding $\uparrow : W^{\mathrm{op}} \to [W, 2]$ given by $w \mapsto \{v \mid w \sqsubseteq v\}$ preserves meets and exponentials.
- An element is a **prime** ($p \sqsubseteq \bigsqcup_i d_i \Rightarrow \exists i.\ p \sqsubseteq d_i$) iff it is $\uparrow w$.
- Every upper set S is the join of primes below it:

$$S = \bigsqcup \{P \mid P \text{ prime}, P \subseteq S\} = \bigsqcup \{\uparrow w \mid w \in S\}$$

In short: $[W, 2]$ is a **prime algebraic lattice** [Win09].

A **duality** (Raney [Ran52]; Nielsen, Plotkin, and Winskel [NPW81]):

$$\mathbf{Pos}^{\mathrm{op}} \simeq \mathbf{PrAlgLatt}$$

Dualities and modalities

The main duality is now

$$\mathbf{Stable}^{\mathrm{op}} \simeq \mathbf{Coh}$$

between

- **distributive lattices** and stable (= $\wedge$-preserving) maps
- **coherent frames** and Scott-continuous, $\bigsqcap$-preserving maps (**not** the usual category from Stone duality)

Then

> The stable semantics and the Heyting algebra semantics are **equi-complete**, **constructively**.

All previous work on modalities carries through, nearly verbatim.

Categorifying the stable semantics

Let $\mathcal{C}$ be a category with finite products and coproducts, which is also a **co-distributive category**: $a + (c \times d) \cong (a + c) \times (a + d)$.

> A two-dimensional stable semantics is a semantics of proofs in a **category of algebras** over a co-distributive theory.

Why? Because 'filters' are **product-preserving presheaves** over $\mathcal{C}$!

If $\mathcal{C}$ is a **Lawvere theory**, then the product-preserving presheaves $[\mathcal{C}, \mathbf{Set}]_\times \cong \mathrm{Sind}(\mathcal{C}^{\mathrm{op}})$ are the **algebras of the theory** $\mathcal{C}$.

Fact $\mathcal{C}$ is co-distributive iff $[\mathcal{C}, \mathbf{Set}]_\times$ is cartesian closed.

For any bi-ccc $\mathcal{C}$ we have a bi-ccc functor $\mathcal{C} \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]_\times$. Hence

Theorem

The category $[\mathcal{C}, \mathbf{Set}]_\times$ of product-preserving presheaves over a co-distributive $\mathcal{C}$ is complete for typed λ-calculus with sums.

References I

[Clo18] Ranald Clouston. “Fitch-Style Modal Lambda Calculi”. In: *Foundations of Software Science and Computation Structures*. Ed. by Christel Baier and Ugo Dal Lago. Vol. 10803. Lecture Notes in Computer Science. Cham: Springer International Publishing, 2018, pp. 258–275. DOI: 10.1007/978-3-319-89366-2_14 (cit. on pp. 6–10).

[DM23] Anupam Das and Sonia Marin. “On Intuitionistic Diamonds (and Lack Thereof)”. In: *Automated Reasoning with Analytic Tableaux and Related Methods*. Ed. by Revantha Ramanayake and Josef Urban. Vol. 14278. Lecture Notes in Computer Science. Cham: Springer Nature Switzerland, 2023, pp. 283–301. DOI: 10.1007/978-3-031-43513-3_16 (cit. on p. 24).

References II

[DJK10] Wojciech Dzik, Jouni Järvinen, and Michiro Kondo. "Intuitionistic propositional logic with Galois connections". In: *Logic Journal of IGPL* 18.6 (2010), pp. 837–858. DOI: 10.1093/jigpal/jzp057 (cit. on pp. 27, 30).

[Gv24] Mai Gehrke and Sam van Gool. *Topological Duality for Distributive Lattices: Theory and Applications.* Cambridge Tracts in Theoretical Computer Science 61. Cambridge University Press, 2024. URL: http://arxiv.org/abs/2203.03286 (cit. on p. 30).

References III

[Law94] F William Lawvere. “Tools for the Advancement of Objective Logic: Closed Categories and Toposes”. In: *The Logical Foundations of Cognition*. Ed. by John Macnamara and Gonzalo E Reyes. Oxford University Press, 1994, pp. 43–56. DOI: 10.1093/oso/9780195092158.003.0004 (cit. on pp. 6–10).

[Law73] F. William Lawvere. “Metric spaces, generalized logic, and closed categories”. In: *Rendiconti del Seminario Matematico e Fisico di Milano* 43.1 (1973), pp. 135–166. DOI: 10.1007/BF02924844 (cit. on p. 20).

[LS09] F. William Lawvere and Stephen H. Schanuel. *Conceptual Mathematics: A First Introduction to Categories*. 2nd ed. Cambridge University Press, 2009. DOI: 10.1017/CBO9780511804199 (cit. on pp. 6–10).

References IV

[NPW81] Mogens Nielsen, Gordon Plotkin, and Glynn Winskel. "Petri nets, event structures and domains, Part I". In: *Theoretical Computer Science* 13.1 (1981), pp. 85–108. DOI: 10.1016/0304-3975(81)90112-2 (cit. on pp. 15, 21, 33).

[Ran52] George N. Raney. "Completely distributive complete lattices". In: *Proceedings of the American Mathematical Society* 3.5 (1952), pp. 677–680. DOI: 10.1090/S0002-9939-1952-0052392-3 (cit. on pp. 15, 21, 33).

[Sim94] Alex K. Simpson. "The Proof Theory and Semantics of Intuitionistic Modal Logic". PhD thesis. The University of Edinburgh, 1994. URL: http://hdl.handle.net/1842/407 (cit. on p. 24).

[Win09] Glynn Winskel. "Prime algebraicity". In: *Theoretical Computer Science* 410.41 (2009), pp. 4160–4168. DOI: 10.1016/j.tcs.2009.06.015 (cit. on pp. 15, 21, 33).