Paraconsistent arithmetic and recapture

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Background and motivation

Technical preliminaries

Primitive recursion

Main results

What is recapture?

Beall (2013b): if we are interested in mathematical consequences of a non-classical theory (assuming we believe there are no mathematical dialetheias) we want the non-classical theory to be as strong as the classical one.

How to make sure that this is so?

How to assess how strong non-classical (especially paraconsistent) theories of arithmetic are?

What is recapture?

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Classical recapture for arithmetic

- Beall (2013a): we can add "shrieking rules"

$$\varphi, \neg\varphi \vdash \bot,$$

  where $\bot$ is a sentence entailing triviality, and where φ is a formula of the language under consideration which is assumed to behave classically.

- Friedman and Meyer (1992): in the relevant arithmetic $\mathsf{R}^{\#\#}$ we can recover the theorems of classical Peano Arithmetic PA, *in the classical, i.e. arrow-free, language*.

Objections to recapture strategies

- Halbach and Nicolai (2018): when we add a truth predicate to the language, we lose inductive strength in the non-classical case because of the deductive weakness of non-classical logic;
- Nicolai (2022): strategies similar to "shrieking rules" either require the assumption of shrieking for the whole language (and hence effectively going back to classical logic) or fail when moving to theories of truth/membership/property instantiation;
- Objection to Friedman-Meyer recapture: the meaning of classical connectives, and hence of the theorems of PA which are recovered, is not preserved in the relevant logic (because the material conditional loses its intended meaning).

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Recapture in paraconsistent settings

How to dispel these criticisms: *proof-theoretic analysis* of paraconsistent arithmetics.

Pros:

- The analysis is conducted in the full language of the paraconsistent logic, thus dispelling Friedman-Meyer objections.
- Objective measure, which can be easily adapted to extended languages, possibly dispelling Halbach-Nicolai objections.

Cons:

- Need to formulate ordinal notations via recursion – unclear how to do this in a paraconsistent setting.

Aside: Priest on arithmetical dialetheia

Priest (2006, ch. 17) argues that we can in fact formulate – in the usual way – primitive recursive functions behaving paraconsistently, and uses this as evidence for arithmetical dialetheia.

Heuristically: since we can formulate, via inconsistent primitive recursion, a Gödel sentence which behaves like a dialetheia, we can argue that there are arithmetical dialetheia.

Choi (2022): the notion of primitive recursion in a paraconsistent setting loses its intended meaning – we should be able to reformulate primitive recursion in a paraconsistent-friendly way.

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subDLQ

$$\mathcal{L}_{\texttt{subDLQ}} ::= x = y \mid \bot \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \neg\varphi \mid \varphi \rightarrow \psi \mid \varphi \Rightarrow \psi \mid \forall x \varphi$$

Roughly: logic obtained by supplementing LP with a relevant and a linear conditional.

- $\Rightarrow$ does not contrapose but satisfies the deduction theorem;
- $\rightarrow$ does not weaken but is used to deal with substitutions of identicals;
- The logic satisfies weakening but not contraction (however it satisfies a form of deduction for theorems;
- Explosive negation can be defined as $\sim A := \neg A \Rightarrow \bot$.

Hilbert-style calculus for subDLQ

1. $\bot \to \varphi$
2. $\varphi \to \varphi$
3. $(\varphi \to \psi) \wedge (\psi \to \chi) \to (\varphi \to \chi)$
4. $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$
5. $(\varphi \wedge \psi) \to \varphi$
6. $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
7. $\varphi \to (\varphi \vee \psi)$
8. $\psi \to (\varphi \vee \psi)$
9. $\varphi \wedge (\psi \wedge \chi) \to (\varphi \wedge \psi) \wedge \chi$
10. $\varphi \vee \neg\varphi$
11. $\neg\neg\varphi \leftrightarrow \varphi$
12. $\varphi \vee \psi \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$
13. $\varphi \wedge \psi \leftrightarrow \neg(\neg\varphi \vee \neg\psi)$
14. $\varphi \wedge (\psi \vee \chi) \leftrightarrow (\varphi \wedge \chi) \vee (\varphi \wedge \chi)$
15. $\forall x\varphi \leftrightarrow \neg\exists x\neg\varphi$
16. $\exists x\varphi \leftrightarrow \neg\forall x\neg\varphi$
17. $\forall x\varphi \to \varphi(x/t)$ for any t
18. $\forall x(\varphi \vee \psi) \to (\varphi \vee \forall x\psi)$ for x not free in φ
19. $(\varphi \to \psi) \Rightarrow (\varphi \Rightarrow \psi)$
20. $\neg(\varphi \Rightarrow \psi) \Rightarrow \neg(\varphi \to \psi)$
21. $(\varphi \wedge \neg\psi) \Rightarrow \neg(\varphi \Rightarrow \psi)$
22. $(\psi \Rightarrow \chi) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))$
23. $(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow (\psi \Rightarrow (\varphi \Rightarrow \chi))$
24. $\varphi \Rightarrow (\psi \Rightarrow \varphi)$
25. $(\varphi \Rightarrow \chi) \Rightarrow ((\psi \Rightarrow \chi) \Rightarrow (\varphi \vee \psi \Rightarrow \chi))$
26. $\varphi \Rightarrow (\psi \Rightarrow \varphi \wedge \psi)$
27. $(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow (\varphi \wedge \psi \Rightarrow \chi)$
28. $\forall x(\varphi \Rightarrow \psi) \Rightarrow (\exists y\varphi(x/y) \to \psi)$ for x not free in ψ
29. $\forall x(\psi \Rightarrow \varphi) \Rightarrow (\psi \Rightarrow \forall y\varphi(x/y))$ for x not free in ψ
30. $\forall x(\varphi(x) \wedge \psi(x)) \Rightarrow (\forall x\varphi(x) \wedge \forall x\psi(x))$
31. $x = y \Rightarrow \varphi(x) \to \varphi(y)$
32. $(\varphi \leftrightarrow \psi) \Rightarrow \chi(\varphi) \leftrightarrow \chi(\psi)$.

Rules:

$$\frac{\varphi \qquad \varphi \Rightarrow \psi}{\psi}\ \mathsf{MP} \qquad\qquad \frac{\varphi}{\forall x \varphi}\ \forall\mathsf{I}$$

Definition

A formula φ of $\mathcal{L}_{\texttt{subDLQ}}$ is said to be derivable form a finite multiset of formulae Γ, in symbols $\Gamma \vdash \varphi$, if there is a sequence of formulae $\varphi_0, \ldots, \varphi_n$ such that $\varphi_n = \varphi$ and for every φ_i, either $\varphi_i \in \Gamma$, or φ_i is an instance of an axiom of subDLQ, or results from an application of a rule on previous lines that have not been used in other applications or rules.

The subDLQ consequence relation satisfies the following:

- $\Gamma, \varphi \vdash \varphi$;
- If $\Gamma \vdash \varphi$ and $\Delta, \varphi \vdash \psi$, then $\Gamma, \Delta \vdash \psi$;
- If $\Gamma, \varphi \vdash \psi$ then $\Gamma, \varphi, \chi \vdash \psi$ (weakening);
- If $\Gamma, \Delta \vdash \varphi$, then $\Delta, \Gamma \vdash \varphi$ (interchange).[1]

[1] Trivially valid since Γ and Δ are multisets.

subDLQ – A

"The goals for this chapter are more staid – to develop a more 'classical' arithmetic, providing some reassurance. No contradictions about the natural numbers are proved, and the attitude is neutral as to whether arithmetic may ultimately be inconsistent or not" (Weber 2021, p. 197)

I. $0 = sx \Rightarrow \bot$;

II. $sx = sy \Rightarrow x = y$;

III. $x + 0 = x$
$x + sy = s(x + y)$;

IV. $x \times 0 = 0$
$x \times sy = (x \times y) + x$;

V. $\varphi(0) \wedge \forall x(\varphi(x) \Rightarrow \varphi(sx)) \Rightarrow \forall x \varphi(x)$.

$$\forall x(\neg\varphi(x) \Rightarrow \exists y(y < x \wedge \neg\varphi(y))) \Rightarrow \forall x \varphi(x) \quad \text{(VI)}$$

$$\exists x \exists y \varphi(x, y) \Rightarrow \exists x \exists y(\varphi(x, y) \wedge \forall u \forall v(\varphi(u, v) \Rightarrow x \leqslant u \wedge y \leqslant v)) \quad \text{(VII)}$$

$$\forall x \forall y((y < x \Rightarrow \varphi(y)) \Rightarrow \varphi(x)) \Rightarrow \forall x \varphi(x) \quad \text{(VIII)}$$

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Validities

- Addition is commutative and associative;
- Multiplication is commutative, associative and distributes over addition;
- Exponentiation is definable as usual and satisfies usual properties;
- Order is definable as follows:

$$x \leqslant y := \exists n(x + n = y)$$
$$x < y := \exists n(x + sn = y)$$

  and satisfies the following properties

$$x \leqslant y \Rightarrow x = y \vee x < y$$
$$\forall x(0 \leqslant x) \qquad \forall x(0 < sx)$$
$$\forall x \forall y(x \leqslant x + y) \qquad \forall x \forall y(x < x + sy)$$
$$\forall x(x < 0 \Rightarrow \bot) \qquad \forall x(x < sx)$$

  $\leqslant$ is a partial order, while $<$ is a strict partial order.
- The numbers are totally ordered.

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Division and primes

- $\mathtt{div}(x, y) := \exists n(xn = y)$;
- $\mathtt{gcd}(x, y, t) := \mathtt{div}(t, x) \wedge \mathtt{div}(t, y) \wedge \forall u(\mathtt{div}(u, x) \wedge \mathtt{div}(u, y) \Rightarrow u \leqslant t)$;
- $\mathtt{Prime}(p) := \forall x(\neg\mathtt{div}(ssx, p) \vee ssx = p) \vee p = 1$.

- The smallest divisor of n is prime.
- If p is prime and $\mathtt{div}(p, xy)$ then $\mathtt{div}(p, x)$ or $\mathtt{div}(p, y)$ or $p \neq p$.
- $\mathtt{div}(p, \prod_{i=0}^{n} x_i) \Rightarrow \bigvee_{i=0}^{n} \mathtt{div}(p, x_i)$ or $p \neq p$, for p prime.

Proposition (Fundamental theorem of arithmetic, Weber 2021)

Let $n > 1$. Then there are primes $p_0, \ldots, p_m$ such that

$$n = \prod_{i=0}^{m} p_i$$

is unique up to inconsistency: for any other such $q_0, \ldots, q_l$, either each p_i is identical to exactly one q_j, or some $p_i \neq p_i$.[a]

[a] The proof of this claim relies on complete induction.

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How to address paraconsistent recursion

- Primitive recursive functions: can be effectively computed by an algorithmic procedure, i.e. output is computable by performing a finite number of steps given a basic set of instructions of finite size.
- Inconsistent settings: the output may not be unique, since – for instance – it could be a non-self-identical number.
- This entails that decoding cannot be defined as in the classical case.
- Possible solution: defining a notion of primitive recursion that is compatible with the inconsistent case.

Primitive recursive mappings

A relation $f\colon X \longrightarrow Y$ is a mapping iff:

- $f \sqsubseteq X \times Y$;
- $\forall x(x \in X \Rightarrow \exists y(y \in Y \wedge \langle x, y\rangle \in f))$;
- $\langle x, y\rangle \in f \Rightarrow (z \neq y \Rightarrow \langle x, z\rangle \notin f)$

Primitive recursive *mappings* can be defined as usual. Also characteristic mappings will need to be adapted to inconsistent case.

$$g(x_1, \ldots, x_n) \ni \begin{cases} 1 & \text{iff } F(x_1, \ldots, x_n) \\ 0 & \text{iff } \neg F(x_1, \ldots, x_n) \end{cases}$$

Accordingly we can define an ordinal notation up to ε_0 following Pohlers (2009) almost directly.

Ordinal notations

- The notion of ordinal behind our coding is classical, therefore it satisfies Cantor's Normal Form theorem.
- Since addition and exponentiation satisfy the same properties as the classical ones in $\mathsf{subDLQ} - \mathsf{A}$, CNF theorem holds also for ordinal notations.
- Everything is formulated in a paraconsistent logic, so we cannot be guaranteed that there isn't some inconsistency in the result of some arithmetical operation.

Decoding

Whenever we perform an operation on ordinal notations, we obviously want the result to transfer to the ordinals they are notations of. *This is not necessarily the case in a paraconsistent setting.*

Let $\alpha, \eta, \xi \in \mathtt{OT}$, and $\alpha = \eta + \xi$. In a classical setting, this should be equivalent to $|\alpha| = |\eta| + |\xi|$.

- $\alpha = \langle \alpha_1, \ldots, \alpha_n \rangle = \prod_{i=0}^{n} \mathtt{Pnb}(i)^{x_1+1}$.
- Let, for some $0 \leqslant i \leqslant n$, $\mathtt{Pnb}(i) \neq \mathtt{Pnb}(i)$.
- Since $\alpha \neq \alpha$, $\alpha = \eta + \xi$ and $\alpha \neq \eta + \xi$.

How to solve this: introduce an axiom negating the possibility that $x \neq x$.

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Back to recapture

Given the fundamental theorem of arithmetic in subDLQ − A, we have to assume (Ref) to obtain a unique decoding.

Hence we effectively are assuming that the arithmetic portion of the language satisfies explosion, i.e. behaves classically. Is this the same thing as Beall's shrieking?

Not really: as we will see, the recapture results we will show hold for extensions of the language with arbitrary predicates (like a truth predicate). Hence we are not assuming explosion for the whole language.

We have something like the kind of recapture suggested by Fiore and Rosenblatt (2023): the strength of the classical theory is recovered by assuming classicality only for a relevant portion of the language, namely that appearing in some of the assumptions of our proof (more on this later).

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Lower bound: proof strategy

What we show: subDLQ − A, with the classical identity axiom, proves transfinite induction for any ordinal less than ϵ_0 for all formulae of the language of subDLQ − A *possibly expanded by finitely many predicates which behave classically or paraconsistently.*

This is showed by simply adapting to `subDLQ` Gentzen's proof, in the version presented by Fischer et al. for another non-classical logic, HYPE.

Useful definitions:

- $\texttt{Prog}(A) := \forall \eta \forall \xi (\xi \prec \eta \Rightarrow A(\xi)) \Rightarrow A(\eta)$.
- $A^{+}(\theta) := \forall \xi (\forall \eta (\eta \prec \xi \Rightarrow A(\eta)) \Rightarrow \forall \eta (\eta \prec \xi + \omega^{\theta} \Rightarrow A(\eta)))$.

The usual suspects

Proposition

For $\theta \in \mathtt{OT}$, $\mathrm{subDLQ} - \mathrm{A} \vdash \theta = 0 \vee \theta \succ 0$.

Lemma

$\vdash \mathit{Prog}(A) \Rightarrow \mathit{Prog}(A^{+})$.

Lemma

If $TI_{\alpha}(\mathcal{L}^{+}_{subDLQ})$ *is derivable in* $\mathrm{subDLQ} - \mathrm{A}$, *then* $TI_{\omega^{\alpha}}(\mathcal{L}^{+}_{subDLQ})$ *is derivable in* $\mathrm{subDLQ} - \mathrm{A}$.

Theorem

For all $\alpha \prec \varepsilon_0$, $\mathrm{subDLQ} - \mathrm{A} \vdash \mathit{Prog}(A) \Rightarrow \forall \xi \prec \alpha A(\xi)$, **with** A **being any predicate of** $\mathcal{L}^{+}_{subDLQ}$.

A few things to highlight

$\texttt{Prog}(A) \Rightarrow \texttt{Prog}(A^+)$:

- Assume $\Gamma := \{\texttt{Prog}(A), \forall\zeta(\zeta \prec \theta \rightarrow A^+(\zeta)), \forall\zeta(\zeta \prec \xi \Rightarrow A(\xi)), \eta \prec \xi + \omega^\theta\}$[2]. Work towards establishing $A(\eta)$. We will show:

$$\Gamma \vdash \theta = 0 \Rightarrow A(\eta) \qquad (\dagger)$$
$$\Gamma \vdash \theta \succ 0 \Rightarrow A(\eta) \qquad (\ddagger)$$

- To show ($\ddagger$), we need an instance of CNF for ordinal notations in subDLQ − A. Hence classical identity for unique decoding is fundamental.
- All other steps follow Fischer et al. (2023) adapting the proof to `subDLQ`, with minor modifications in the proof strategy to avoid contraction.

Rest of the proof: always following Fischer et al. adapting the proof to `subDLQ`.

[2]Where Γ is a multiset.

Discussion

What does this tell us about recapture for paraconsistent arithmetics?

- If we presuppose classical identity, the amount of transfinite induction for the extended language is the same as Peano Arithmetic.
  - Dispelling the objection in Halbach and Nicolai (2018);
  - Analogous result to that for HYPE (Fischer et al. 2021) but for a paraconsistent substructural logic.
- We are not making the language fully classical (because the result holds for an extended language), hence we don't have to "shriek" the whole language: dispelling the objection in Nicolai (2022).
- Still, we have to presuppose classical identity for the whole of the arithmetical language to obtain the result.
- We also have to strengthen the axiomatisation with extra induction axioms to be able to define the coding.
- Bonus point: undermining Priest's argument in favour of arithmetical dialetheia based on a dialethic Gödel sentence.

Many thanks!

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