Topological semantics of the predicate modal calculus QGL extended with non-well-founded proofs

Daniyar Shamkanov [1,2]
Joint work with Pavel Razumnyy [2]

[1]Steklov Mathematical Institute of Russian Academy of Sciences
Moscow

[2]National Research University Higher School of Economics
Moscow

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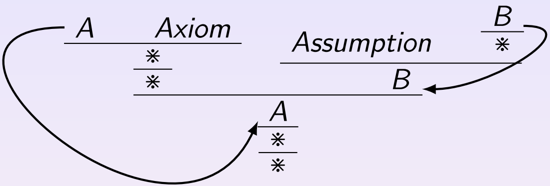
Cyclic and non-well-founded derivations

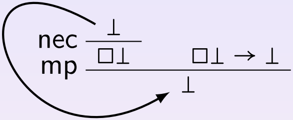


Deductive systems allowing cyclic and non-well-founded derivations can be defined for

- modal μ-calculus,
- action logic,
- Peano arithmetic,
- GL, Grz, K^+, etc.

Cyclic derivations in GL

Example of a cyclic derivation



Theorem

GL = K4 + cyclic derivations, i.e.,

$$\Gamma \vdash_{\mathsf{GL}} A \Longleftrightarrow \Gamma \vdash_{\mathsf{cycl}} A$$

Non-well-founded derivations in GL

Definition

An *∞-derivation in* GL is a (possibly infinite) tree whose nodes are marked by modal formulas and that is constructed according to the rules (mp) and (nec). In addition, any infinite branch in an ∞-derivation must contain infinitely many applications of the rule (nec).

Example

$$\text{mp}\,\dfrac{\text{nec}\,\dfrac{\text{mp}\,\dfrac{\text{nec}\,\dfrac{\text{mp}\,\dfrac{\overset{\vdots}{\Box p_3} \qquad \Box p_3 \to p_2}{p_2}}{\Box p_2} \qquad \Box p_2 \to p_1}{p_1}}{\Box p_1} \qquad \Box p_1 \to p_0}{p_0}$$

Non-well-founded derivations in GL

Example

$$
\text{mp}\,\dfrac{\text{nec}\,\dfrac{\text{mp}\,\dfrac{\text{nec}\,\dfrac{\text{mp}\,\dfrac{\overset{\vdots}{\Box p_3} \qquad \Box p_3 \to p_2}{p_2}}{\Box p_2} \qquad \Box p_2 \to p_1}{p_1}}{\Box p_1} \qquad \Box p_1 \to p_0}{p_0}
$$

Definition

An *assumption leaf* of an ∞-derivation is a leaf that is not marked by an axiom of K4.

Proposition

If a formula A is provable by an ∞-proof, then $\mathsf{GL} \vdash A$.

Global neighbourhood completeness of GL

Definition

We set $\Gamma \vdash_\infty A$ if there is an ∞-derivation with the root marked by A in which all assumption leafs are marked by some elements of Γ.

Definition

We set $\Gamma \vDash_g A$ if for any neighbourhood GL-model $\mathcal{M}$

$$(\forall B \in \Gamma \;\; \mathcal{M} \vDash B) \Longrightarrow \mathcal{M} \vDash A.$$

Theorem

$\Gamma \vdash_\infty A \Longleftrightarrow \Gamma \vDash_g A.$

The predicate version of GL

A predicate modal calculus QGL

Let us fix a first-order signature without function symbols and constants.

Axioms

- axioms of GL
- $\forall x\, A(x) \to A(y)$
- $\forall x\, (A \to B) \to (A \to \forall x\, B)$, where $x \notin FV(A)$

Inference rules

$$\text{mp}\ \frac{A \qquad A \to B}{B} \qquad \text{nec}\ \frac{A}{\Box A} \qquad \text{gen}\ \frac{A}{\forall x\, A}$$

Theorem (Montagna 1984)

The calculus QGL is not arithmetically complete. Besides, it is not complete with respect to its Kripke semantics.

Theorem (Vardanyan 1985)

The quantified provability logic of PA (denoted by QPL(PA)) is Π^0_2-complete.

Theorem (Borges and Joosten 2023)

The strictly positive fragments of QPL(PA) and QGL coincide and are equal to QRC_1.

Back to Kripke semantics of QGL

Montagna's counterexample

The formula

$$\forall x\, \exists y\, \Box(\Box P(y) \to P(x)) \to \forall x\, \Box P(x)$$

is valid in any relational QGL-frame, but is not provable in QGL.

Non-well-founded derivations in QGL

Definition

An *∞-derivation* in QGL is a (possibly infinite) tree whose nodes are marked by predicate modal formulas and that is constructed according to the rules (mp), (gen) and (nec). In addition, any infinite branch in an ∞-derivation must contain infinitely many applications of the rule (nec).

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \\ \Box\forall x_2\, P_2(x_2)\end{matrix} \qquad \Box\forall x_2\, P_2(x_2) \to P_1(x_1)}{P_1(x_1)}\,\text{mp}}{\forall x_1\, P_1(x_1)}\,\text{gen}}{\Box\forall x_1\, P_1(x_1)}\,\text{nec} \qquad \Box\forall x_1\, P_1(x_1) \to P_0(x_0)}{P_0(x_0)}\,\text{mp}$$

Topological completeness

Definition

An *∞-proof* is an ∞-derivation, where all leaves are marked by axioms of QGL. We set $\mathsf{QGL}_\infty \vdash A$ if there is an ∞-proof with the root marked by A.

Theorem (topological completeness)

For any formula A, $\mathsf{QGL}_\infty \vdash A$ if and only if A is valid in every predicate topological frame of QGL_∞.

Observation

We have

$$\mathsf{QGL}_\infty \nvdash \forall x\, \exists y\, \Box(\Box P(y) \to P(x)) \to \forall x\, \Box P(x).$$

Hence, QGL_∞ is not complete for its relational interpretation.

Strong local completeness

Definition

We put $\Gamma \vdash A$ if there is an ∞-derivation δ with the root marked by A such that, for each leaf a of δ that is not marked by an axiom, a is marked by a formula from Γ, and there are no applications of the rules (gen) and (nec) on the path from the root of δ to the leaf a.

Notice that $\Gamma \vdash A$ if and only if $\mathsf{QGL}_\infty \vdash \bigwedge \Gamma_0 \to A$ for some finite subset Γ_0 of Γ.

Theorem (strong local completeness)

We have $\Gamma \vdash A$ if and only if A is a local semantic consequence of Γ over the class of predicate topological frames of QGL_∞.

Weak topological completeness

Topological semantics

A *predicate topological frame (for* QGL_∞*)* is a tuple (X, τ, D), where (X, τ) is a scattered topological space and D is a non-empty domain.

A *valuation in D* is a function sending each n-ary predicate letter to an n-ary relation on D, and a *variable assignment* is a function from the set of variables $Var = \{x_0, x_1, x_2, \dots\}$ to the domain D.

A *predicate topological model* $\mathcal{M} = (X, \tau, D, \xi)$ is a predicate topological frame (X, τ, D) together with an indexed family of valuations $\xi = (\xi_w)_{w \in X}$ in D.

Topological semantics

The *truth of a formula A at a world w of a model $\mathcal{M} = (X, \tau, D, \xi)$ under a variable assignment h* is defined as

- $\mathcal{M}, w, h \not\models \bot$,
- $\mathcal{M}, w, h \models P(x_1, \ldots, x_n) \iff (h(x_1), \ldots, h(x_n)) \in \xi_w(P)$,
- $\mathcal{M}, w, h \models A \to B \iff \mathcal{M}, w, h \not\models A$ or $\mathcal{M}, w, h \models B$,
- $\mathcal{M}, w, h \models \Box A \iff \exists U \in \tau\, (w \in U$ and $\forall w' \in U \setminus \{w\}\ \mathcal{M}, w', h \models A)$,
- $\mathcal{M}, w, h \models \forall x\, A \iff \mathcal{M}, w, h' \models A$ for any varible assignment h' such that $h' \overset{x}{=} h$,

where $h' \overset{x}{=} h$ means that $h'(y) = h(y)$ for each $y \in Var \setminus \{x\}$.

A *formula A is true in $\mathcal{M}$* if A is true at all worlds of $\mathcal{M}$ under all variable assignments. In addition, *A is valid in a frame $\mathcal{F}$* if A is true in all models over $\mathcal{F}$.

Example

The Barcan formula

$$\forall x \ \Box P(x) \to \Box \forall x \, P(x)$$

is not valid in this topological semantics while domains are constant.

Topological completeness via a sequent calculus

Lemma (soundness)

If $\mathsf{QGL}_\infty \vdash A$, then A is valid in every predicate topological frame.

The converse direction:

1. consider a sequent calculus for QGL_∞ with non-well-founded proofs;
2. combine sequent-based completeness proofs for the classical predicate calculus and for GL extended with non-well-founded proofs.

A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas. For a finite multiset of formulas $\Gamma = B_1, \ldots, B_n$, we set $\Box\Gamma := \Box B_1, \ldots, \Box B_n$.

Non-well-founded sequent calculus

Initial sequents and inference rules of the sequent calculus S have the following form:

$$\Gamma, P(\vec{x}) \Rightarrow P(\vec{x}), \Delta, \qquad \Gamma, \bot \Rightarrow \Delta,$$

$$\to_{\mathsf{L}} \frac{\Gamma, B \Rightarrow \Delta \qquad \Gamma \Rightarrow A, \Delta}{\Gamma, A \to B \Rightarrow \Delta}, \qquad \to_{\mathsf{R}} \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta},$$

$$\forall_{\mathsf{L}} \frac{\Gamma, A(y), \forall x\, A \Rightarrow \Delta}{\Gamma, \forall x\, A \Rightarrow \Delta}, \qquad \forall_{\mathsf{R}} \frac{\Gamma \Rightarrow A(y), \Delta}{\Gamma \Rightarrow \forall x\, A, \Delta}\,(y \notin FV(\Gamma \cup \Delta)),$$

$$\Box \frac{\Gamma, \Box\Gamma \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A, \Delta}.$$

Every infinite branch in a non-well-founded proof of this calculus must contain infinitely many applications of the rule $(\Box)$.

Example

$$
\rightarrow_L \frac{\dfrac{\text{Ax}}{P, \Box(\Box P \to P) \Rightarrow P} \qquad \Box\,\dfrac{\Box P \to P, \Box(\Box P \to P) \Rightarrow P}{\Box(\Box P \to P) \Rightarrow P, \Box P}}{\Box\,\dfrac{\Box P \to P, \Box(\Box P \to P) \Rightarrow P}{\Box(\Box P \to P) \Rightarrow \Box P}}
$$

The proof of completeness

Observation

If A is valid in every predicate topological frame, then A is true in any predicate topological model with a countable domain under any bijective variable assignment.

Lemma

If a formula $\bigwedge \Gamma \rightarrow \bigvee \Delta$ is true in in any predicate topological model with a countable domain under any bijective variable assignment, then the sequent $\Gamma \Rightarrow \Delta$ is provable in S.

Lemma

If a sequent $\Gamma \Rightarrow \Delta$ is provable in S, then $\mathrm{QGL}_{\infty} \vdash \bigwedge \Gamma \rightarrow \bigvee \Delta$.

Corollary (completeness)

For any formula A, $\mathrm{QGL}_{\infty} \vdash A$ if and only if A is valid in every predicate topological frame.

Strong topological completeness

Shehtman's ultrabouquet construction

Ultrabouquet of topological spaces

For any $i \in I$, let $\mathcal{X}_i = (X_i, \tau_i)$ be a topological space and w_i be a closed point in it. Let $\mathcal{U}$ be a non-principal ultrafilter in I. The *ultrabouquet* $\bigvee_{\mathcal{U}}(\mathcal{X}_i, w_i)$ is a topological space obtained as a set from the disjoint union $\bigsqcup_{i \in I} X_i$ by identifying all points w_i. A set *U is open in* $\bigvee_{\mathcal{U}}(\mathcal{X}_i, w_i)$ if and only if

- the set $U \cap (X_i \setminus \{w_i\})$ is open in $\mathcal{X}_i$ for any $i \in I$,
- $\{i \in I \mid U \cap X_i \text{ is open in } \mathcal{X}_i\} \in \mathcal{U}$ whenever $w_* \in U$,

where w_* is the point of $\bigvee_{\mathcal{U}}(\mathcal{X}_i, w_i)$ obtained by identifying points w_i.

An ultrabouquet of scattered topological spaces is a scattered topological space.

The case of predicate models

Ultrabouquet of predicate topological models

For $i \in I$, let $\mathcal{M}_i = (X_i, \tau_i, D_i, \xi_i)$ be a predicate topological model, w_i be a closed point in it and h_i be a variable assignment in $\mathcal{M}_i$. Given a non-principal ultrafilter $\mathcal{U}$ in I, we define $\bigvee_{\mathcal{U}}(\mathcal{M}_i, w_i)$ as a tuple (X, τ, D, ξ), where $(X, \tau) = \bigvee_{\mathcal{U}}((X_i, \tau_i), w_i)$ and $D = \prod_{i \in I} D_i$. In addition, $(a_1, \ldots, a_n) \in \xi_w(P)$ if and only if

- $(\pi_i(a_1), \ldots, \pi_i(a_n)) \in (\xi_i)_w(P)$ for $w \in X_i \smallsetminus \{w_i\}$,
- $\{i \in I \mid (\pi_i(a_1), \ldots, \pi_i(a_n)) \in (\xi_i)_w(P)\} \in \mathcal{U}$ whenever $w = w_*$.

Besides, we define the variable assignment $h\colon Var \to D$ so that $\pi_i \circ h = h_i$ for any $i \in I$.

Topological compactness

Definition

We set $\Gamma \vDash A$ if for any predicate topological model $\mathcal{M} = (X, \tau, D, \xi)$, any world w of $\mathcal{M}$ and any variable assignment $h\colon Var \to D$

$$\forall B \in \Gamma\ \mathcal{M}, w, h \vDash B \Longrightarrow \mathcal{M}, w, h \vDash A.$$

Theorem

If $\Gamma \vDash A$, then there is a finite subset Γ_0 of Γ such that $\Gamma_0 \vDash A$.

Strong topological completeness of QGL_∞

Theorem (strong local completeness)

For any set of formulas Γ and any formula A,

$$\Gamma \vdash A \iff \Gamma \vDash A.$$

Proof of the right-to-left implication:

If $\Gamma \vDash A$, then $\Gamma_0 \vDash A$ for some finite subset Γ_0 of Γ. Therefore, $\bigwedge \Gamma_0 \to A$ is valid in every predicate topological frame and $\mathsf{QGL}_\infty \vdash \bigwedge \Gamma_0 \to A$. It follows that $\Gamma \vdash A$.

Topological incompleteness of QGL

Proposition

The set of theorems of QGL_∞ is not computably enumerable

(this has been checked for a signature with function symbols).

Corollary

The system QGL is not complete with respect to its topological semantics.

Conjecture

The set of theorems of QGL_∞ is Σ^1_1-complete.

Connections with arithmetic

Question

Doesn't QGL_∞ coincide with the first-order provability logic of $\mathsf{ACA}_0 + \{\text{true } \Sigma^1_2\text{-sentences}\}$ (or $\mathsf{ACA}_0 + \{\text{true } \Sigma^1_3\text{-sentences}\}$)?

Remark

For every pair of true Π^1_1-sentences A and B, one of them implies the other over $\mathsf{ACA}_0 + \{\text{true } \Sigma^1_1\text{-sentences}\}$.

Therefore, the provability logic of $\mathsf{ACA}_0 + \{\text{true } \Sigma^1_1\text{-sentences}\}$ contains some form of linearity.

Theorem (Aguilera and Pakhomov, forthcoming)

In ZFC + «there are infinitely many Woodin cardinals», there is pair of true Π^1_{2n}-sentences (Σ^1_{2n+1}-sentences) that are mutually independent over the theory $\mathsf{ACA}_0 + \{\text{true } \Sigma^1_{2n}\text{-sentences}\}$ ($\mathsf{ACA}_0 + \{\text{true } \Pi^1_{2n+1}\text{-sentences}\}$).

Some other avenues for further research:

- In the context of categorical logic, I wonder what representation theorem underlies completeness of QGL_∞.
- It seems that theorems of QGL_∞ obtained by computable ∞-proofs belong to $\mathsf{QPL}(\mathsf{PA} + \{\text{true } \Pi^0_2\text{-sentences}\})$. What else can be said about the fragment of QGL_∞ with only computable ∞-proofs?

Thank you!