Atomization alternatives in the Russell-Prawitz translation

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Polymorphic Lambda Calculus / System F

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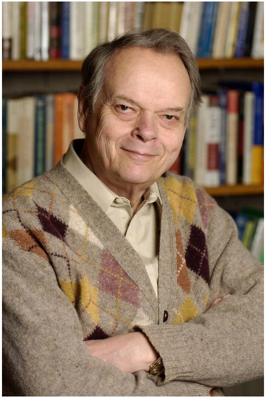


John C. Reynolds

Jean-Yves Girard

System **F** (for logicians)

Natural deduction style

Formulas $X \mid A \wedge B \mid A \to B \mid \forall X.A$

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ A & B\end{array}}{A \wedge B}\ \wedge\mathrm{I} \qquad \dfrac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \to B}\ \to\mathrm{I} \qquad \dfrac{\begin{array}{c}\vdots \\ A\end{array}}{\forall X.A}\ \forall\mathrm{I}$$

$$\dfrac{\begin{array}{c}\vdots \\ A \wedge B\end{array}}{A}\ \wedge\mathrm{E} \qquad \dfrac{\begin{array}{cc}\vdots & \vdots \\ A \to B & A\end{array}}{B}\ \to\mathrm{E} \qquad \dfrac{\begin{array}{c}\vdots \\ \forall X.A\end{array}}{A[F/X]}\ \forall\mathrm{E}$$

F any formula

System **F** (for computer scientists)

λ-calculus style

Types $\quad X \mid A \wedge B \mid A \to B \mid \forall X.A$

Terms $\quad x \mid t1 \mid t2 \mid \langle t, s \rangle \mid t\,s \mid \lambda x^A.t \mid t\,F \mid \Lambda X.t$

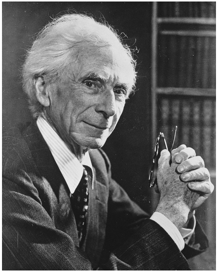
$$\frac{\Gamma \vdash t : A \wedge B}{\Gamma \vdash t1 : A} \qquad \frac{\Gamma \vdash t : A \to B \qquad \Gamma \vdash s : A}{\Gamma \vdash t\,s : B} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B}$$

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash s : B}{\Gamma \vdash \langle t, s \rangle : A \wedge B} \qquad \frac{\Gamma \vdash t : A}{\Gamma \vdash \Lambda X.t : \forall X.A} \qquad \frac{\Gamma \vdash t : \forall X.A}{\Gamma \vdash t\,F : A[F/X]}$$

F any type

Russell-Prawitz translation

Russell-Prawitz translation



Bertrand Russell

Dag Prawitz

Russell-Prawitz translation

IPC	$\hookrightarrow$	**F**
$\wedge, \rightarrow, \bot, \vee$		$\wedge, \rightarrow, \forall$

Russell-Prawitz translation

$$\begin{array}{ccc} & (\cdot)^{RP} & \\ \textbf{IPC} & \hookrightarrow & \textbf{F} \\ \wedge, \rightarrow, \bot, \vee & & \wedge, \rightarrow, \forall \end{array}$$

Russell-Prawitz's translation of formulas:

Russell-Prawitz translation

$$\begin{array}{ccc} & (\cdot)^{RP} & \\ \textbf{IPC} & \hookrightarrow & \textbf{F} \\ \wedge, \to, \bot, \vee & & \wedge, \to, \forall \end{array}$$

Russell-Prawitz's translation of formulas:

$$X^* :\equiv X$$
$$(A \wedge B)^* :\equiv A^* \wedge B^*$$
$$(A \to B)^* :\equiv A^* \to B^*$$
$$\bot^* :\equiv \forall X.X$$
$$(A \vee B)^* :\equiv \forall X.((A^* \to X) \wedge (B^* \to X)) \to X.$$

Russell-Prawitz translation

$$\begin{array}{ccc} & (\cdot)^{RP} & \\ \textbf{IPC} & \hookrightarrow & \textbf{F} \\ \wedge, \rightarrow, \bot, \vee & & \wedge, \rightarrow, \forall \end{array}$$

RP-translation of formulas + RP-translation of proofs

Translation of **IPC** into **F**

$$\dfrac{\begin{array}{c} \vdots \\ A \vee B \end{array} \quad \begin{array}{c} [A] \\ \vdots \\ F \end{array} \quad \begin{array}{c} [B] \\ \vdots \\ F \end{array}}{F} \vee E$$

Translation of **IPC** into **F**

$$\dfrac{\begin{array}{c}\vdots\\ A \vee B\end{array} \qquad \begin{array}{c}[A]\\ \vdots\\ F\end{array} \qquad \begin{array}{c}[B]\\ \vdots\\ F\end{array}}{F}\ \vee\mathrm{E}$$

In system **F**:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ (A \vee B)^* :\equiv \forall X.((A^* \to X) \wedge (B^* \to X)) \to X\end{array}}{((A^* \to F^*) \wedge (B^* \to F^*)) \to F^*} \qquad \dfrac{\dfrac{\begin{array}{c}[A^*]\\ \vdots\\ F^*\end{array}}{A^* \to F^*} \quad \dfrac{\begin{array}{c}[B^*]\\ \vdots\\ F^*\end{array}}{B^* \to F^*}}{(A^* \to F^*) \wedge (B^* \to F^*)}}{F^*}$$

Translation of **IPC** into **F**

$$\begin{array}{ccc} \mathbf{IPC} & \overset{(\cdot)^{RP}}{\hookrightarrow} & \mathbf{F} \\ \wedge, \to, \bot, \vee & & \wedge, \to, \forall \end{array}$$

RP-translation of formulas + RP-translation of proofs

Translation of **IPC** into **F**

$$\begin{array}{ccc} & (\cdot)^{RP} & \\ \textbf{IPC} & \hookrightarrow & \textbf{F} \\ \wedge, \rightarrow, \perp, \vee & & \wedge, \rightarrow, \forall \end{array}$$

• impredicative system

RP-translation of formulas + RP-translation of proofs

Translation of **IPC** into **F**

$$\begin{array}{ccc} \textbf{IPC} & \overset{(\cdot)^{RP}}{\hookrightarrow} & \textbf{F} \\ \wedge, \to, \bot, \vee & & \wedge, \to, \forall \end{array}$$

- impredicative system
- no identity preservation

RP-translation of formulas + RP-translation of proofs

β-conversions

β-conversions

$$\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \to B} \qquad \begin{matrix}\vdots \\ A\end{matrix}}{B} \quad \rightsquigarrow \quad \begin{matrix}\vdots \\ A \\ \vdots \\ B\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{A \vee B} \qquad \begin{matrix}[A]\\ \vdots \\ F\end{matrix} \qquad \begin{matrix}[B]\\ \vdots \\ F\end{matrix}}{F} \quad \rightsquigarrow \quad \begin{matrix}\vdots \\ A \\ \vdots \\ F\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{\forall X.A}}{A[F/X]} \quad \rightsquigarrow \quad \begin{matrix}\vdots \\ A[F/X]\end{matrix}$$

η-conversions

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ A \to B\end{matrix} \qquad A}{B}}{A \to B} \quad \rightsquigarrow \quad \begin{matrix}\vdots \\ A \to B\end{matrix}$$

$$\dfrac{\begin{matrix}\vdots \\ A \vee B\end{matrix} \qquad \dfrac{[A]}{A \vee B} \qquad \dfrac{[B]}{A \vee B}}{A \vee B} \quad \rightsquigarrow \quad \begin{matrix}\vdots \\ A \vee B\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \forall X.A\end{matrix}}{A}}{\forall X.A} \quad \rightsquigarrow \quad \begin{matrix}\vdots \\ \forall X.A\end{matrix}$$

Commuting conversions for **IPC**

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \bot\end{array}}{F} \qquad \vdots}{D}\,\mathsf{r} \quad \overset{\gamma}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\vdots\\ \bot\end{array}}{\mathrm{D}}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A \vee B\end{array} \quad \begin{array}{c}[A]\\ \vdots\\ F\end{array} \quad \begin{array}{c}[B]\\ \vdots\\ F\end{array}}{F} \qquad \vdots}{D}\,\mathsf{r} \quad \overset{\gamma}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\vdots\\ A \vee B\end{array} \quad \dfrac{\begin{array}{c}[A]\\ \vdots\\ F\end{array} \quad \vdots}{D}\,\mathsf{r} \quad \dfrac{\begin{array}{c}[B]\\ \vdots\\ F\end{array} \quad \vdots}{D}\,\mathsf{r}}{D}$$

No identity perservation

$$\mathbf{IPC} \quad \overset{(\cdot)^{RP}}{\hookrightarrow} \quad \mathbf{F}$$

No identity perservation

IPC	$\overset{(\cdot)^{RP}}{\hookrightarrow}$	**F**
β, η, γ		β, η

No identity perservation

$$\begin{array}{ccc} & (\cdot)^{RP} & \\ \textbf{IPC} & \hookrightarrow & \textbf{F} \\ \beta, \eta, \gamma & & \beta, \eta \end{array}$$

$t \rightsquigarrow_\beta q$ $\qquad$ $t^{RP} \rightsquigarrow_\beta^+ q^{RP}$

- preserves β-reductions

No identity perservation

	$(\cdot)^{RP}$		
IPC	$\hookrightarrow$	**F**	
β, η, γ		β, η	
$t \rightsquigarrow_{\beta} q$		$t^{RP} \rightsquigarrow_{\beta}^{+} q^{RP}$	• preserves β-reductions
$t \rightsquigarrow_{\eta} q$		$t^{RP} \not\rightsquigarrow_{\beta\eta} q^{RP}$ $t^{RP} \not\leftrightsquigarrow_{\beta\eta} q^{RP}$	• no strict simulation no identity for η-reductions

No identity perservation

$$\begin{array}{ccc} \textbf{IPC} & \overset{(\cdot)^{RP}}{\hookrightarrow} & \textbf{F} \\ \beta, \eta, \gamma & & \beta, \eta \end{array}$$

$t \rightsquigarrow_\beta q$ $\quad$ $t^{RP} \rightsquigarrow_\beta^+ q^{RP}$ $\quad$ • preserves β-reductions

$t \rightsquigarrow_\eta q$ $\quad$ $t^{RP} \not\rightsquigarrow_{\beta\eta} q^{RP}$, $t^{RP} \not\leftrightsquigarrow_{\beta\eta} q^{RP}$ $\quad$ • no strict simulation
no identity for η-reductions

$t \rightsquigarrow_\gamma q$ $\quad$ $t^{RP} \not\rightsquigarrow_{\beta\eta} q^{RP}$, $t^{RP} \not\leftrightsquigarrow_{\beta\eta} q^{RP}$ $\quad$ • no strict simulation
no identity for cc

Identity perservation

Identity perservation

$$\mathbf{IPC} \quad \overset{(\cdot)^\circ}{\hookrightarrow} \quad \mathbf{F_{at}} \subseteq \mathbf{F}$$

$\mathbf{F_{at}}$ is predicative

Identity perservation

IPC β, η, γ	$\overset{(\cdot)^\circ}{\hookrightarrow}$	$\mathbf{F_{at}} \subseteq \mathbf{F}$ β, η	$\mathbf{F_{at}}$ is predicative
$t \rightsquigarrow_\beta q$		$t^\circ \rightsquigarrow_{\beta\eta} q^\circ$	• preserves β-reductions

Identity perservation

	$(\cdot)^\circ$		
IPC	$\hookrightarrow$	$\mathbf{F_{at}} \subseteq \mathbf{F}$	$\mathbf{F_{at}}$ is predicative
β, η, γ		β, η	
$t \rightsquigarrow_\beta q$		$t^\circ \rightsquigarrow_{\beta\eta} q^\circ$	• preserves β-reductions
$t \rightsquigarrow_\eta q$		$t^\circ \rightsquigarrow_{\beta\eta} q^\circ$	• preserves η-reductions

Identity perservation

IPC β, η, γ	$\overset{(\cdot)^\circ}{\hookrightarrow}$	$\mathbf{F_{at}} \subseteq \mathbf{F}$ β, η	$\mathbf{F_{at}}$ is predicative
$t \rightsquigarrow_\beta q$		$t^\circ \rightsquigarrow_{\beta\eta} q^\circ$	• preserves β-reductions
$t \rightsquigarrow_\eta q$		$t^\circ \rightsquigarrow_{\beta\eta} q^\circ$	• preserves η-reductions
$t \rightsquigarrow_\gamma q$		$t^\circ \not\rightsquigarrow_{\beta\eta} q^\circ$ $t^\circ \leftrightsquigarrow_{\beta\eta} q^\circ$	• no strict simulation but identity for cc

System $\mathbf{F_{at}}$: atomic polymorphism

Formulas $X \mid A \wedge B \mid A \to B \mid \forall X.A$

$$\frac{\begin{array}{cc} \vdots & \vdots \\ A & B \end{array}}{A \wedge B}\ \wedge I \qquad \frac{\begin{array}{c} [A] \\ \vdots \\ B \end{array}}{A \to B}\ \to I \qquad \frac{\begin{array}{c} \vdots \\ A \end{array}}{\forall X.A}\ \forall I$$

$$\frac{\begin{array}{c} \vdots \\ A \wedge B \end{array}}{A}\ \wedge E \qquad \frac{\begin{array}{cc} \vdots & \vdots \\ A \to B & A \end{array}}{B}\ \to E \qquad \frac{\begin{array}{c} \vdots \\ \forall X.A \end{array}}{A[Y/X]}\ \forall E$$

Y atomic

$\mathbf{F}_{at}$ versus $\mathbf{F}$

$\mathbf{F}_{at}$	$\mathbf{F}$
Predicative	Impredicative
Subformula property	No notion of subformula
Embeds **IPC**	Embeds **IPC**
Easy strong normalization proof (Tait's method of reducibility)	Intricate strong normalization proof (Reducibility candidates)
Elementary normalization proof	No elementary normalization proof
Functions provably total in $\lambda^{\rightarrow}$	Functions provably total in PA_2

Embedding of **IPC** into $\mathbf{F_{at}}$

$$\begin{array}{ccc} \mathbf{IPC} & \hookrightarrow & \mathbf{F_{at}} \\ \bot, \wedge, \vee, \to & & \wedge, \to, \forall \end{array}$$

Embedding of **IPC** into $\mathbf{F_{at}}$

$$\begin{array}{ccc} \mathbf{IPC} & \hookrightarrow & \mathbf{F_{at}} \\ \bot, \wedge, \vee, \to & & \wedge, \to, \forall \end{array}$$

"The elimination rules [⊥, ∨] are very bad. What is catastrophic about them is the parasitic presence of a formula F which has no structural link with the formula which is eliminated."

— J.-Y. Girard, *Proofs and Types*, 1989, pages 73-74

Embedding of **IPC** into $\mathbf{F_{at}}$

$$\begin{array}{ccc} \mathbf{IPC} & \hookrightarrow & \mathbf{F_{at}} \\ \bot, \wedge, \vee, \to & & \wedge, \to, \forall \end{array}$$

"The elimination rules [⊥, ∨] are very bad. What is catastrophic about them is the parasitic presence of a formula F which has no structural link with the formula which is eliminated."

— J.-Y. Girard, *Proofs and Types*, 1989, pages 73-74

"One tends to think that natural deduction should be modified to correct such atrocities... It does not seem that the (⊥, ∨) fragment of the calculus is etched on tablets of stone."

— J.-Y. Girard, *Proofs and Types*, 1989, page 80

Embeddings of **IPC** into $\mathbf{F_{at}}$

Embeddings of **IPC** into $\mathbf{F_{at}}$

Russell-Prawitz translation of formulas + different translation of proofs

Embeddings of **IPC** into $\mathbf{F_{at}}$

Russell-Prawitz translation of formulas	+	different translation of proofs

- $\approx$ 2006 - The original embedding $(\cdot)^{\circ}$
  based on instantiation overflow [F. Ferreira]

Embeddings of **IPC** into $\mathbf{F_{at}}$

Russell-Prawitz translation of formulas + different translation of proofs

- $\approx$ 2006 - The original embedding $(\cdot)^{\circ}$
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- $\approx$ 2020 - The embedding $(\cdot)^{\sharp}$
  based on "compact" instantiation overflow [P. Pistone, L. Tranchini, M. Petrolo]

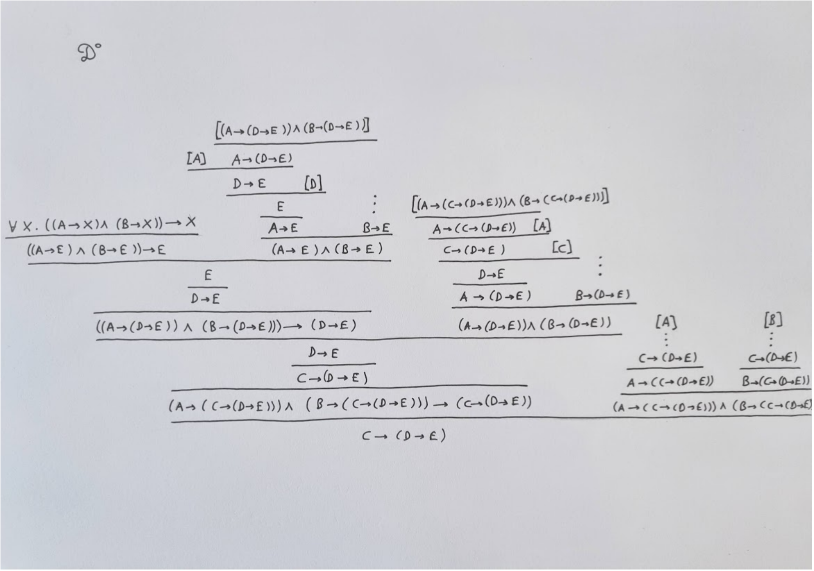
Embeddings of **IPC** into $\mathbf{F_{at}}$

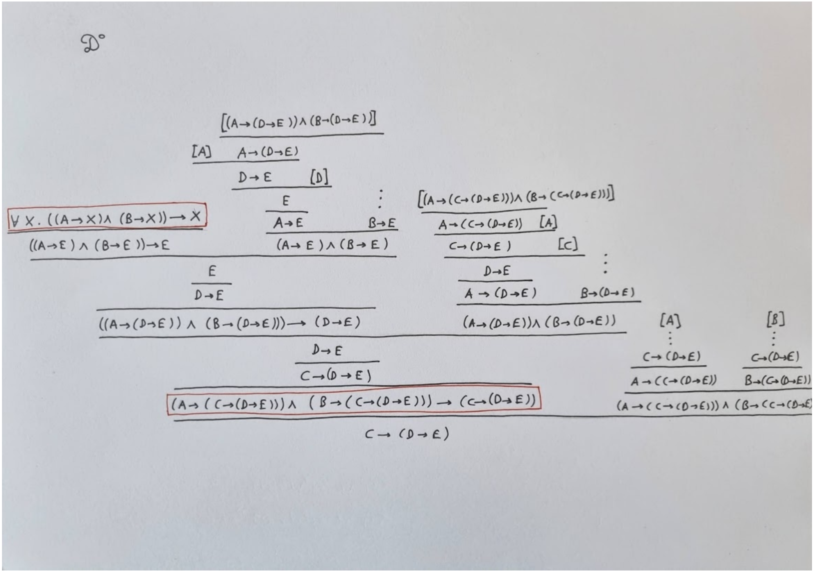
Russell-Prawitz translation of formulas + different translation of proofs

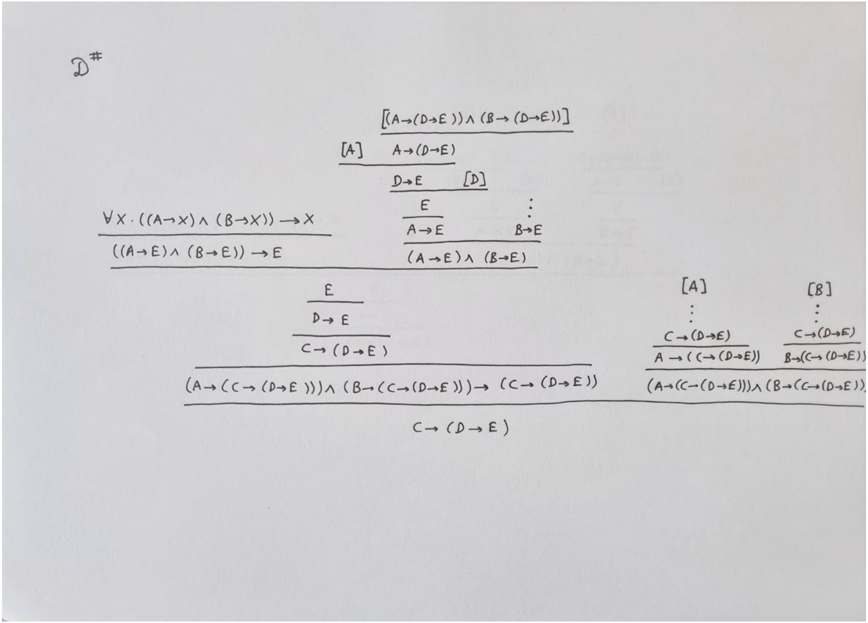
- $\approx$ 2006 - The original embedding $(\cdot)^{\circ}$
  based on instantiation overflow [F. Ferreira]
- $\approx$ 2020 - The embedding $(\cdot)^{\sharp}$
  based on "compact" instantiation overflow [P. Pistone, L. Tranchini, M. Petrolo]
- $\approx$ 2020 - The embedding $(\cdot)^{*}$
  based on admissibility [J. Espírito Santo, G. Ferreira]

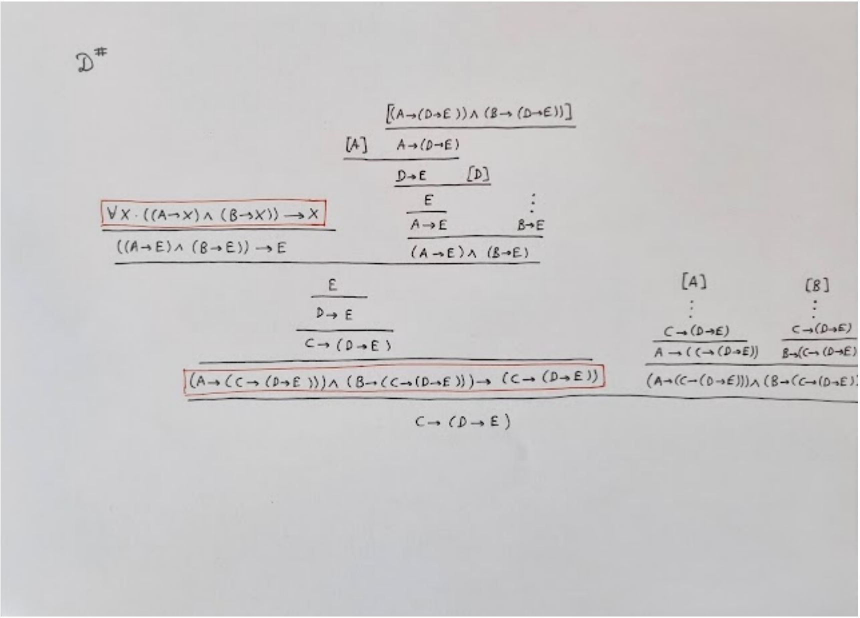
Let $\mathfrak{D}$ be the following derivation in **IPC**:

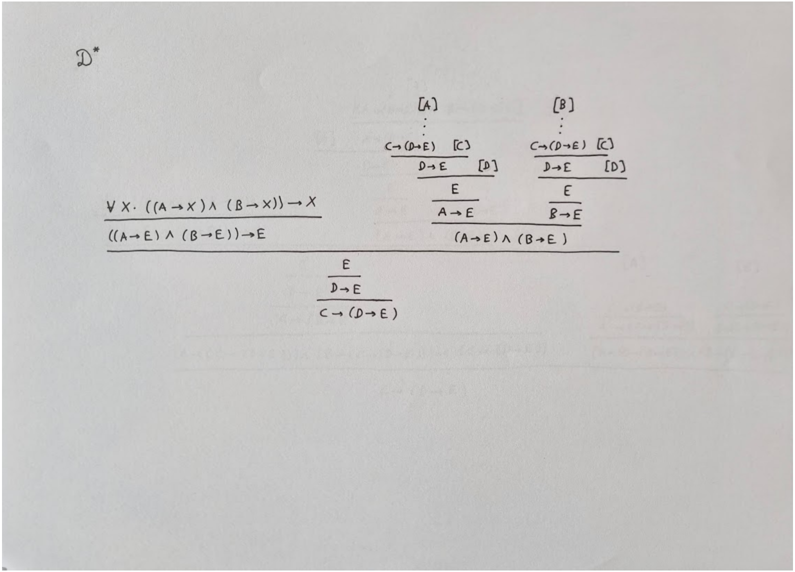
$$\dfrac{A \vee B \qquad \begin{array}{c}[A]\\ \vdots\\ C \to (D \to E)\end{array} \qquad \begin{array}{c}[B]\\ \vdots\\ C \to (D \to E)\end{array}}{C \to (D \to E)}$$


𝒟°
[(A→(D→E))∧(B→(D→E))]
[A] A→(D→E)
D→E [D]
E
A→E B→E
∀X.((A→X)∧(B→X))→X
((A→E)∧(B→E))→E
(A→E)∧(B→E)
[(A→(C→(D→E)))∧(B→(C→(D→E)))]
A→(C→(D→E)) [A]
C→(D→E) [C]
D→E
A→(D→E) B→(D→E)
((A→(D→E))∧(B→(D→E)))→(D→E)
(A→(D→E))∧(B→(D→E))
[A] [B]
C→(D→E) C→(D→E)
A→(C→(D→E)) B→(C→(D→E))
(A→(C→(D→E)))∧(B→(C→(D→E)))→(C→(D→E))
C→(D→E)


𝒟°
[(A→(D→E))∧(B→(D→E))]
[A] A→(D→E)
D→E [D]
E
A→E
B→E
∀X.((A→X)∧(B→X))→X
((A→E)∧(B→E))→E
(A→E)∧(B→E)
E
D→E
((A→(D→E))∧(B→(D→E)))→(D→E)
[(A→(C→(D→E)))∧(B→(C→(D→E)))]
A→(C→(D→E)) [A]
C→(D→E) [C]
D→E
A→(D→E)
B→(D→E)
(A→(D→E))∧(B→(D→E))
D→E
C→(D→E)
(A→(C→(D→E)))∧(B→(C→(D→E)))→(C→(D→E))
[A]
[B]
C→(D→E)
C→(D→E)
A→(C→(D→E))
B→(C→(D→E))
C→(D→E)


$\mathcal{D}^\#$
[(A→(D→E)) ∧ (B→(D→E))]
[A]   A→(D→E)
D→E   [D]
E   ⋮
A→E   B→E
∀X·((A→X) ∧ (B→X)) ⟶ X
((A→E) ∧ (B→E)) ⟶ E   (A→E) ∧ (B→E)
E
D→E
C→(D→E)
[A]   [B]
C→(D→E)   C→(D→E)
A→(C→(D→E))   B→(C→(D→E))
(A→(C→(D→E))) ∧ (B→(C→(D→E))) → (C→(D→E))   (A→(C→(D→E))) ∧ (B→(C→(D→E)))
C→(D→E)


D#
[(A→(D→E))∧(B→(D→E))]
[A] A→(D→E)
D→E [D]
E
A→E B→E
∀X.((A→X)∧(B→X))→X
((A→E)∧(B→E))→E
(A→E)∧(B→E)
E
D→E
C→(D→E)
(A→(C→(D→E)))∧(B→(C→(D→E)))→(C→(D→E))
[A] [B]
C→(D→E) C→(D→E)
A→(C→(D→E)) B→(C→(D→E)
(A→(C→(D→E)))∧(B→(C→(D→E)
C→(D→E)



$\mathcal{D}^*$
[A]
C→(D→E) [C]
D→E [D]
E
A→E
[B]
C→(D→E) [C]
D→E [D]
E
B→E
∀X. ((A→X) ∧ (B→X)) → X
((A→E) ∧ (B→E)) → E
(A→E) ∧ (B→E)
E
D→E
C→(D→E)

Embeddings of IPC into $\mathbf{F_{at}}$

$$\mathbf{IPC} \quad \hookrightarrow \quad \mathbf{F_{at}} \subseteq \mathbf{F}$$

Embeddings of **IPC** into $\mathbf{F_{at}}$

$$\mathbf{IPC} \quad \overset{\overset{\overset{(\cdot)^\circ}{(\cdot)^\sharp}}{(\cdot)^*}}{\hookrightarrow} \quad \mathbf{F_{at}} \subseteq \mathbf{F}$$

Embeddings of **IPC** into $\mathbf{F_{at}}$

$$\mathbf{IPC} \quad \overset{(\cdot)^{*}}{\overset{(\cdot)^{\sharp}}{\overset{(\cdot)^{\circ}}{\hookrightarrow}}} \quad \mathbf{F_{at}} \subseteq \mathbf{F}$$

- All translations work equaly well at the level of provability

Embeddings of **IPC** into $\mathbf{F_{at}}$

$$\mathbf{IPC} \quad \overset{\substack{(\cdot)^{\circ} \\ (\cdot)^{\sharp} \\ (\cdot)^{*}}}{\hookrightarrow} \quad \mathbf{F_{at}} \subseteq \mathbf{F}$$

- All translations work equaly well at the level of provability
- All translations work equaly well at the level of proof identity

Embeddings of **IPC** into $\mathbf{F_{at}}$

$$\mathbf{IPC} \quad \overset{\substack{(\cdot)^{\circ} \\ (\cdot)^{\sharp} \\ (\cdot)^{*}}}{\hookrightarrow} \quad \mathbf{F_{at}} \subseteq \mathbf{F}$$

- All translations work equaly well at the level of provability
- All translations work equaly well at the level of proof identity
- Diferences in the size of derivations and in the size of reduction sequences

Embeddings of **IPC** into $\mathbf{F_{at}}$

$$\mathbf{IPC} \quad \overset{\substack{(\cdot)^\circ \\ (\cdot)^\sharp \\ (\cdot)^*}}{\hookrightarrow} \quad \mathbf{F_{at}} \subseteq \mathbf{F}$$

- All translations work equaly well at the level of provability
- All translations work equaly well at the level of proof identity
- Diferences in the size of derivations and in the size of reduction sequences

$$t^\circ \rightsquigarrow_\beta^+ t^\sharp \rightsquigarrow_\beta^+ t^*$$

Adding atomization conversions

Adding atomization conversions

$$\mathbf{IPC} \quad \overset{(\cdot)^{RP}}{\hookrightarrow} \quad \mathbf{F}$$

Adding atomization conversions

$$\begin{array}{ccc} & (\cdot)^{RP} & \\ \textbf{IPC} & \hookrightarrow & \textbf{F} \\ \beta, \eta, \gamma & & \beta, \eta, \rho \end{array}$$

Adding atomization conversions

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$$\mathbf{IPC} \quad \overset{(\cdot)^{RP}}{\hookrightarrow} \quad \mathbf{F}$$

Adding atomization conversions

$$\begin{array}{ccc} \mathbf{IPC} & \overset{(\cdot)^{RP}}{\hookrightarrow} & \mathbf{F} \\ \beta, \eta, \gamma & & \beta, \eta, \rho \end{array}$$

Adding atomization conversions

$$\begin{array}{ccc} \mathbf{IPC} & \overset{(\cdot)^{RP}}{\hookrightarrow} & \mathbf{F} \\ \beta, \eta, \gamma & & \beta, \eta, \ \rho \end{array}$$

- Atomization conversions do not collapse proof identity

Adding atomization conversions

$$\begin{array}{ccc} & (\cdot)^{RP} & \\ \textbf{IPC} & \hookrightarrow & \textbf{F} \\ \beta, \eta, \gamma & & \beta, \eta, \rho \end{array}$$

- Atomization conversions do not collapse proof identity
- Unique normal form w.r.t. the atomization conversions – *atomic normal form* of the proof

Adding atomization conversions

$$\begin{array}{ccc} & (\cdot)^{RP} & \\ \mathbf{IPC} & \hookrightarrow & \mathbf{F} \\ \beta, \eta, \gamma & & \beta, \eta, \rho \end{array}$$

- Atomization conversions do not collapse proof identity
- Unique normal form w.r.t. the atomization conversions – *atomic normal form* of the proof
- Strict simulation of proof reductions

Adding atomization conversions

$$\begin{array}{ccc} & (\cdot)^{RP} & \\ \textbf{IPC} & \hookrightarrow & \textbf{F} \\ \beta, \eta, \gamma & & \beta, \eta, \rho \end{array}$$

- Atomization conversions do not collapse proof identity
- Unique normal form w.r.t. the atomization conversions – *atomic normal form* of the proof
- Strict simulation of proof reductions

$$t \rightsquigarrow_{\beta\eta\gamma} q \text{ in } \textbf{IPC} \Rightarrow t^{RP} \rightsquigarrow_{\beta\eta\rho} q^{RP} \text{ in } \textbf{F}$$

Adding atomization conversions

$$\begin{array}{ccc} & (\cdot)^{RP} & \\ \mathbf{IPC} & \hookrightarrow & \mathbf{F} \\ \beta, \eta, \gamma & & \beta, \eta, \rho \end{array}$$

- Atomization conversions do not collapse proof identity
- Unique normal form w.r.t. the atomization conversions – *atomic normal form* of the proof
- Strict simulation of proof reductions

$$t \rightsquigarrow_{\beta\eta\gamma} q \text{ in } \mathbf{IPC} \Rightarrow t^{RP} \rightsquigarrow_{\beta\eta\rho} q^{RP} \text{ in } \mathbf{F}$$

- Allows to relate translations into **F** with translations into $\mathbf{F_{at}}$

Recovering the $(\cdot)^*$-translation from the RP-translation

Recovering the $(\cdot)^*$-translation from the RP-translation

IPC

Recovering the $(\cdot)^*$-translation from the RP-translation

IPC

t

Recovering the $(\cdot)^*$-translation from the *RP*-translation

IPC	$\mathbf{F_{at}}$
t	t^*

Recovering the $(\cdot)^*$-translation from the RP-translation

IPC	$\mathbf{F_{at}}$	**F**
t	t^*	t^{RP}

Recovering the $(\cdot)^*$-translation from the RP-translation

IPC	$\mathbf{F_{at}}$		**F**
t	t^*	$\leftsquigarrow_\rho$	t^{RP}

Recovering the $(\cdot)^*$-translation from the RP-translation

IPC	$\mathbf{F_{at}}$		**F**	
t	t^*	$\leftsquigarrow_\rho$	t^{RP}	atomic normal form

Recovering the $(\cdot)^*$-translation from the RP-translation

IPC	$\mathbf{F_{at}}$		**F**	
t	t^*	$\leftsquigarrow_\rho$	t^{RP}	atomic normal form
$\rightsquigarrow_{\beta\eta\gamma}$				
q				

Recovering the $(\cdot)^*$-translation from the RP-translation

IPC	$\mathbf{F_{at}}$		**F**	
t	t^*	$\leftsquigarrow_{\rho}$	t^{RP}	atomic normal form
$\rightsquigarrow_{\beta\eta\gamma}$			$\rightsquigarrow_{\beta\eta\rho}$	
q			q^{RP}	
				strict simulation

Recovering the $(\cdot)^*$-translation from the RP-translation

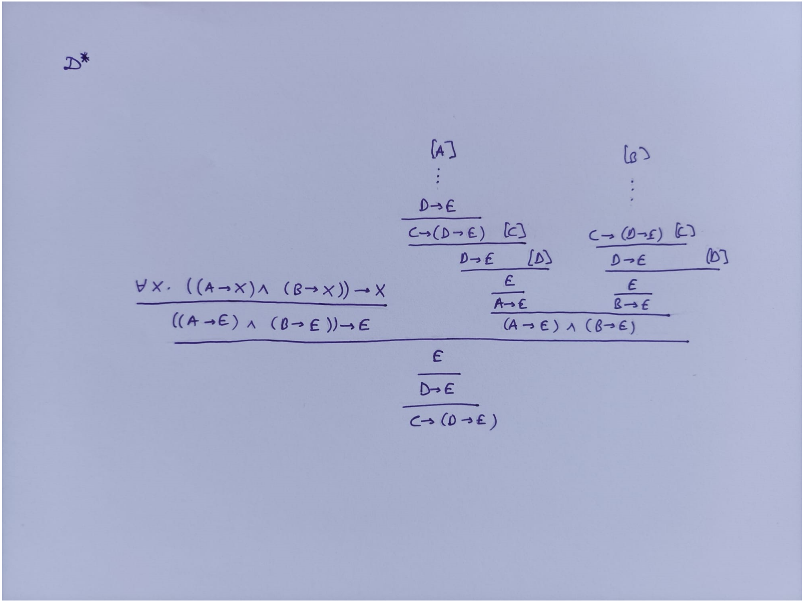
IPC	$\mathbf{F_{at}}$		**F**	
t	t^*	$\leftsquigarrow_\rho$	t^{RP}	atomic normal form
$\downarrow_{\beta\eta\gamma}$	$\updownarrow_{\beta\eta}$		$\downarrow_{\beta\eta\rho}$	
q	q^*	$\leftsquigarrow_\rho$	q^{RP}	

strict simulation

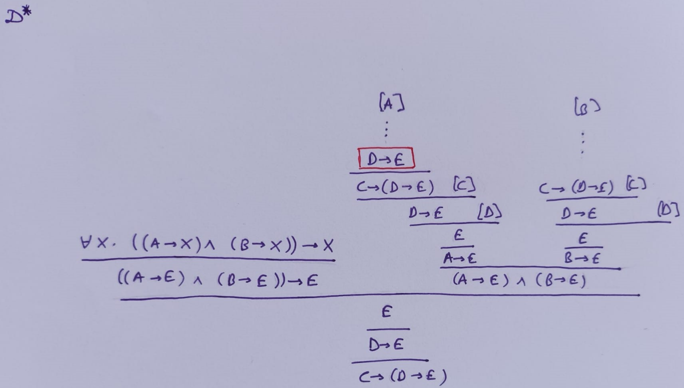
New translation

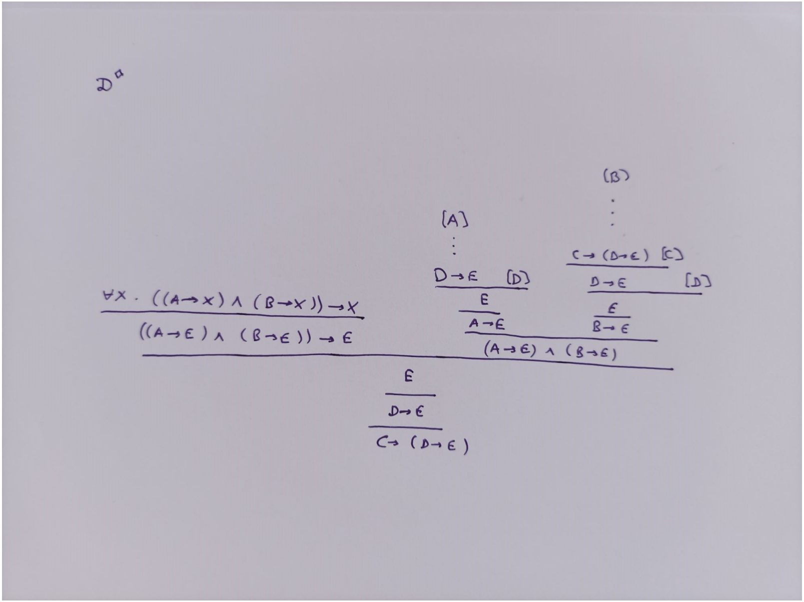
IPC	$\mathbf{F_{at}}$	**F**	
t	$t^{\diamond}$	t^{RP}	atomic normal form
$\rightsquigarrow_{\beta\eta\gamma}$	$\rightsquigarrow_{\beta\eta}$	$\rightsquigarrow_{\beta\eta\rho}$	
q	$q^{\diamond}$	q^{RP}	

strict simulation



$\mathcal{D}^*$
[A]
[B]
D → E
C → (D → E) [C]
C → (D → E) [C]
D → E [D]
D → E [D]
∀X. ((A → X) ∧ (B → X)) → X
E
E
((A → E) ∧ (B → E)) → E
A → E
B → E
(A → E) ∧ (B → E)
E
D → E
C → (D → E)


D*
[A]
D→E
C→(D→E) [C]
D→E [D]
E
A→E
[B]
C→(D→E) [C]
D→E [D]
E
B→E
∀X. ((A→X)∧(B→X))→X
((A→E)∧(B→E))→E
(A→E)∧(B→E)
E
D→E
C→(D→E)


D^α
(B)
[A]
C→(D→E) [C]
D→E [D]
D→E [D]
E
E
A→E
B→E
∀X. ((A→X) ∧ (B→X)) → X
((A→E) ∧ (B→E)) → E
(A→E) ∧ (B→E)
E
D→E
C→(D→E)

Reduction on the fly

$$(MN)^* = M^* N^*$$

Reduction on the fly

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$$(MN)^\diamond = M^\diamond @ N^\diamond$$

Reduction on the fly

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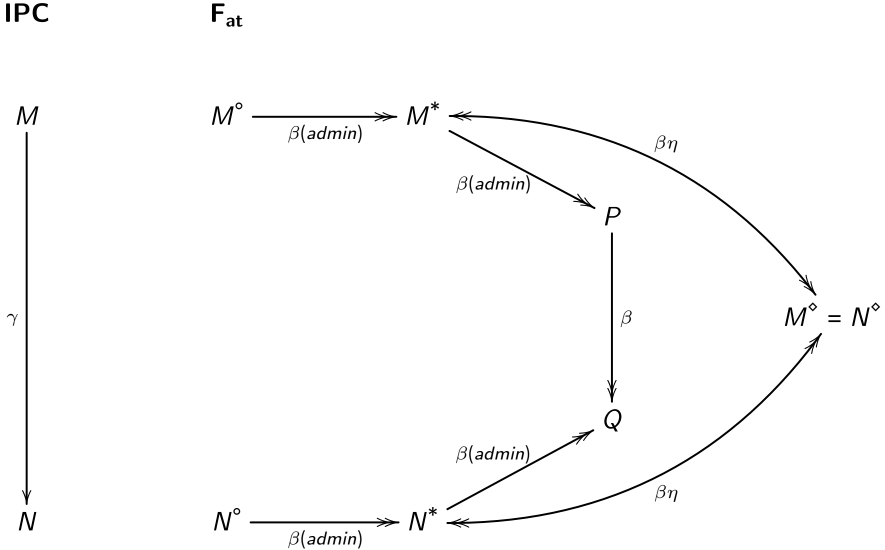
$$(MN)^\diamond = M^\diamond @ N^\diamond$$

$$M@N := \begin{cases} P[N/x] & \text{if } M = \lambda x.P \\ MN & \text{otherwise} \end{cases}$$

Optimized elimination constructions

Commuting conversions



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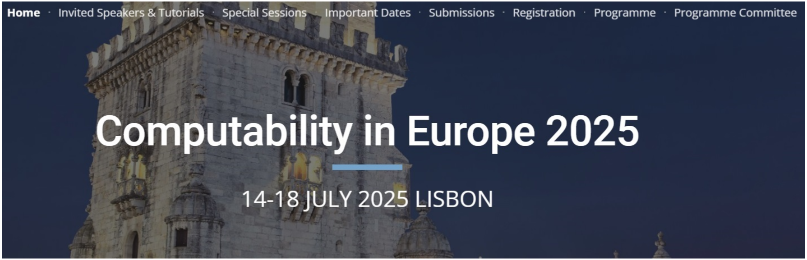
THANK YOU

Invitation



Join us in Lisbon!

Enriching **F** with atomization conversions

Enriching **F** with atomization conversions

The usual $\beta\eta$-conversions:

$$
\begin{array}{llcll}
\beta & (\lambda x.t)q & \rightsquigarrow_{\beta_\rightarrow} & t[q/x] & \\
 & \langle t_1, t_2\rangle i & \rightsquigarrow_{\beta_\wedge} & t_i & (i = 1, 2) \\
 & (\Lambda X.t)B & \rightsquigarrow_{\beta_\forall} & t[B/X] & \\
 & & & & \\
\eta & \lambda x.tx & \rightsquigarrow_{\eta_\rightarrow} & t & (x \notin t) \\
 & \langle t1, t2\rangle & \rightsquigarrow_{\eta_\wedge} & t & \\
 & \Lambda X.tX & \rightsquigarrow_{\eta_\forall} & t & (X \notin t)
\end{array}
$$

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\end{array}
$$

$$
\begin{array}{llcll}
\eta & \lambda x.tx & \rightsquigarrow_{\eta_\rightarrow} & t & (x \notin t) \\
 & \langle t1, t2\rangle & \rightsquigarrow_{\eta_\wedge} & t & \\
 & \Lambda X.tX & \rightsquigarrow_{\eta_\forall} & t & (X \notin t)
\end{array}
$$

The new (atomization) ρ-convertions

$$
\begin{array}{llcl}
\rho & t(C_1 \rightarrow C_2)\langle \lambda x^A.p, \lambda y^B.q\rangle & \rightsquigarrow_\rho & \lambda z^{C_1}.tC_2\langle \lambda x^A.pz, \lambda y^B.qz\rangle \\
 & t(C_1 \wedge C_2)\langle \lambda x^A.p, \lambda y^B.q\rangle & \rightsquigarrow_\rho & \langle tC_i\langle \lambda x^A.pi, \lambda y^B.qi\rangle\rangle_{i=1,2} \\
 & t(\forall Y.D)\langle \lambda x^A.p, \lambda y^B.q\rangle & \rightsquigarrow_\rho & \Lambda Y.tD\langle \lambda x^A.pY, \lambda y^B.qY\rangle
\end{array}
$$