Strong Completeness below GLP

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Everything on earth has its own time and its own season.
Ecclesiastes 3:1

GLP overview

The logic GLP

Logic GLP is the smallest set of formulæ in $\mathcal{L}_\Box$ closed under modus ponens, that contains classical tautologies and modal axioms which reflect provability nature of the Boxes:

1. $[n](p \to q) \to ([n]p \to [n]q)$ (Normality)
2. $[n]([n]p \to p) \to [n]p$ (Löb)
3. $[m]p \to [n][m]p, m \leq n$
4. $\langle m \rangle p \to [n]\langle m \rangle p, m < n$
5. $[m]p \to [n]p, m \leq n$

It is arithmetically complete!

Kripke semantics

We call $(W, <_i)_{i<\omega}$ a Kripke frame if W is a set and $<_i \in W \times W$, then:

1. $[\![p]\!] \subset W$ for $p \in \mathrm{Vars}$;
2. $[\![\langle n \rangle \varphi]\!] = \{x \in W : \exists y \in [\![\varphi]\!](x <_n y)\}$;

GLP is Kripke incomplete.

Topological semantics

Let $(X, \tau_i)_{i<\omega}$ be a (poly)topological space, then

1. $[\![p]\!] \subset W$ for $p \in \text{Vars}$;
2. $[\![\langle n \rangle \varphi]\!] = d_{\tau_n}[\![\varphi]\!]$;

where

$$d_\tau A = \{x : \forall U \in \tau(x \in U \to \exists y \in (A - \{x\}) \cap U)\}$$

is the set of limit points of $A \subset X$.

GLP is topologically complete[1] w.r.t. Beklemishev-Gabelaia (ordinal) spaces[2].

[1] Shamkanov, "Global neighbourhood completeness of the provability logic GLP".
[2] Beklemishev and Gabelaia, "Topological completeness of the provability logic GLP".

General problems of GLP

1. it is 'the' provability logic – arithmetical completeness;
2. it is quite capricious;
3. no Kripke completeness;
4. topological completeness is extremely tricky;
    - we have it (including strong) for topologies in general (Shamkanov);
    - we have it (excluding strong) for ordinal spaces (Beklemishev, Gabelaia)
    - canonical toplogies??? (no strong completeness)

That is why people study its fragments, which are sometimes interesting enough not only to be helpful for modal logic related problems, but also with proof theory, ordinal analysis and so on.

The closed fragment of GLP

Ignatiev logic

Logic I is the smallest set of formulæ in $\mathcal{L}_{\Box}$ closed under modus ponens, that contains classical tautologies and the modal axioms:

1. $[n](p \to q) \to ([n]p \to [n]q)$ (Normality)
2. $[n]([n]p \to p) \to [n]p$ (Löb)
3. $[m]p \to [n][m]p, m \le n$
4. $\langle m \rangle p \to [n]\langle m \rangle p, m < n$
5. ~~$[m]p \to [n]p, m \le n$~~

There are non-trivial Kripke models for this logic.

Some facts

Let $\mathcal{L}_0$ be the variable free (poly)modal language. Then

Fact (Ignatiev)

$\mathsf{GLP} \cap \mathcal{L}_0 = \mathsf{I} \cap \mathcal{L}_0$.

Fact (Ignatiev)

The closed fragmanet of GLP *is Kripke complete, moreover it is Kripke complete w.r.t. a single frame, namely the Ignatiev frame $\mathfrak{I}$ (see next slides). Indeed, it is Kripke complete w.r.t to a designated set of points, namely the main axis.*

On the other hand

Theorem (Strong completeness)

(A) *The closed fragment of* GLP *is strongly complete with respect to* $\mathfrak{I}_{\leq\varepsilon_0}$. *More precisely: let* Γ *be a set of closed* $\mathcal{L}$*-formulæ. Then the following are equivalent:*

  (i) Γ *is consistent with* GLP*; and*

  (ii) $\mathfrak{I}_{\leq\varepsilon_0}, \vec{\alpha} \Vdash \Gamma$ *for some* $\vec{\alpha} \in \mathfrak{I}_{\leq\varepsilon_0}$.

(B) *Moreover,*

  (i) *The closed fragment of* GLP *is not strongly complete with respect to* $\mathfrak{I}_{<\varepsilon_0}$*; and*

  (ii) *The closed fragment of* GLP *is not strongly complete with respect to the main axis of* $\mathfrak{I}_{\leq\varepsilon_0}$.

Here $\mathfrak{I}_{<\varepsilon_0}$

Counterexamples

Lemma

Let $\Gamma = \{\langle 0\rangle^k \top : k < \omega\} \cup \{[1]\bot\} \cup \{[0][1]\bot\}$*. Then,* Γ *is consistent with* GLP*, but for all* $\vec{\alpha} \in \mathrm{ma}(\mathfrak{I}_{\leq\epsilon_0})$*, we have* $\mathfrak{I}_{<\epsilon_0}, \vec{\alpha} \nVdash \Gamma$*. Moreover* Γ *cannot be satisfied at any point of any Icard or Beklemishev-Gabelaia spaces.*

Proof.

It is easy to verify directly that $\mathfrak{I}_{\leq\epsilon_0}, \langle\omega, 0\rangle \Vdash \Gamma$, so that indeed Γ is consistent with GLP. Suppose $\mathfrak{I}_{<\epsilon_0}, \vec{\alpha} \Vdash \Gamma$. Then by $\langle 0\rangle^k\top$, we must have $\alpha_0 > k$. By $[1]\bot$, we must have $\alpha_1 = 0$ and by $[0][1]\bot$ we must have $\alpha_0 \leq \omega$, so the only point in $\mathfrak{I}_{\leq\epsilon_0}$ which satisfies Γ is $\langle\omega, 0\rangle$, which is not on the main axis of $\mathfrak{I}_{\leq\epsilon_0}$. □

The logic J and some topological completeness

Definition (of J)

Logic J is the smallest set of formulæ in $\mathcal{L}_\Box$ closed under modus ponens, that contains classical tautologies and the modal axioms:

1. $[n](p \to q) \to ([n]p \to [n]q)$ (Normality)
2. $[n]([n]p \to p) \to [n]p$ (Löb)
3. $[m]p \to [n][m]p, m \leq n$
4. $\langle m \rangle p \to [n]\langle m \rangle p, m < n$
5. $[m]p \to [m][n]p, m \leq n$;

We have $\mathsf{I} \subsetneq \mathsf{J} \subsetneq \mathsf{GLP}$. There are non-trivial Kripke models for this logic.

Completeness (of J)

It is known to be complete w.r.t. J-frames (due to Beklemishev?). A frame $(X, <_i)_{i<\omega}$ satisfies J if and only if:

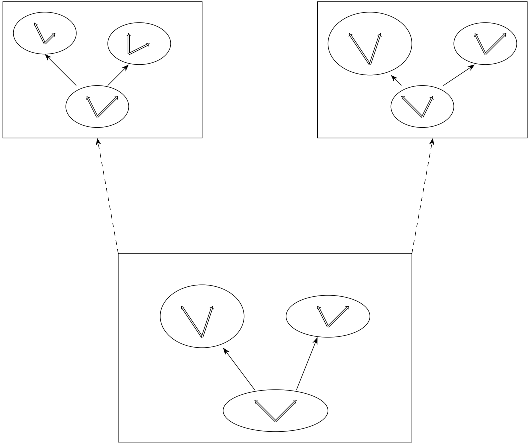
$$\forall x, y, z(x <_n y \to (x <_m z \leftrightarrow y <_m z)),\ m < n;$$

$$\forall x, y, z(x <_m y \wedge y <_n z \to x <_m z),\ m \leq n;$$

Fact (Beklemishev)

J *is sound and complete w.r.t.* J*-frames (inded finite* J*-trees).*

One can observe that the transitive symmetric reflexive closure of $\bigcup_{i>n} <_i$ is an equivalence relation, we call the classes of equivalence n-planes, moreover we if we take two n-planes A and B we have either for each pair $a \in A, b \in B$ we have $a <_n b$ or each such pair is incomparable.



Here $<_0$ represented by dashed arrows, $<_1$ by solid arrows, $<_2$ by double arrows.

Kripke completeness (-ish)

Although GLP is Kripke incomplete, we have completeness in some weaker sense, namely for the J-models who satisfy "enough GLP".

Fact (Beklemishev)

GLP $\vdash \varphi$ *if and only if* $J \vdash M^+(\varphi)$.

where

$$M(\varphi) := \bigwedge_{i<s} \bigwedge_{k=m_i+1}^{n} ([m_i]\,\varphi_i \to [k]\varphi_i)$$

and $\mathrm{sub}(\varphi) = \{\varphi_i : i < s\}$. And $M^+(\varphi) := M(\varphi) \wedge \bigwedge_{m \le n}[m]M(\varphi) \to \varphi$.

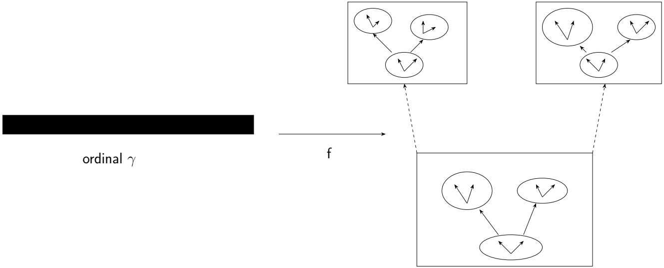
The pullback construction

Generally the standard technique to attain topological completeness for ordinals is the following. Given a non-theorem φ of the logic we find a Kripke frame (-ish) M such that $M, w \Vdash \neg\varphi$, then for some ordinal γ find a map $f : \gamma \to F$ that would preserve enough structure and pull back the valuation i.e. $v_\gamma(p) = f^{-1}[v_F(p)]$ and prove that $f(\alpha) = w$ implies $\alpha \Vdash \neg\varphi$.
E.g. this is a key lemma to achieve topological completeness:

Fact (Beklemishev, Gabelaia)

Let X be a GLP_n-space, T a J_n-tree, $f : X \to T$ a J_n-morphism and φ a $\mathcal{L}_n$-formula. Then $X \vDash \varphi$ iff $T \vDash M^+(\varphi) \to \varphi$.

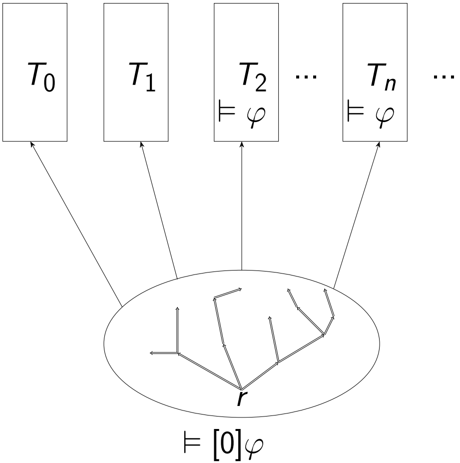
The pullback construction



The strong completeness conundrum

- well, the counterexample :(
- no strong Kripke completeness – fixable with bouquets construction;
- failure of building a rank preserving lifting – fixable by trowing out point we don't like;
- so far works only for GLP $+ [1]^n\bot$;

A typical small J-bouquet



Where T_i is a finite J_2-tree.

The theorem

Lemma (Aguilera, S.)

Given a small J_2 bouquet B with root r such that $r \Vdash [1]^n\bot$ for some n, then there is a countable ordinal γ with BG topologies τ_0, τ_1 and a subspace $G \subset \gamma$ such that there is a j_2-limit-map $f : (G, \tau_0, \tau_1) \to (B, <_0, <_1)$.

Corollary

Let $n \in \mathbb{N}$. Then, $\mathsf{GLP} + [1]^n\bot$ is strongly complete with respect to the class of subspaces of a Beklemishev-Gabelaia space.

To sum up

We have:

- strong Kripke completeness for the closed fragment of GLP;
- no strong completeness of GLP for Icard or Beklemishev-Gabelaia spaces;
- strong completeness for J;
- strong completeness of $\mathrm{GLP} + [1]^{\perp}$ for subspaces of BG-spaces;

Thank you all!