IGL without sharps

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Intuitionistic Gödel-Löb Logic

- GL: $\Box(\Box P \to P) \to \Box P$
- iGL: GL on an intuitionistic base, only boxes
  See [3] for more on iGL.
- IGL: GL on an intuitionistic base, boxes and diamonds
  First developed by Das, van der Giessen and Marin [2]

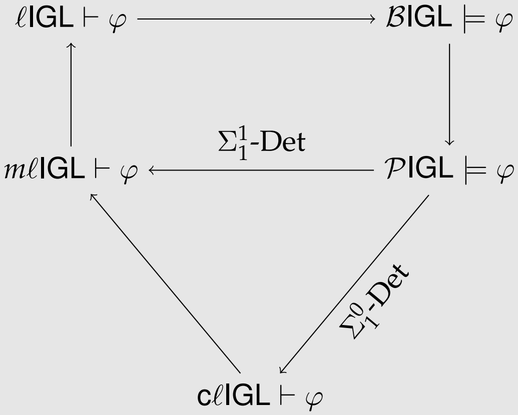
IGL

Das, van der Giessen, and Marin proved:

$$\begin{array}{ccc}
\ell\mathsf{IGL} \vdash \varphi & \longrightarrow & \mathcal{B}\mathsf{IGL} \models \varphi \\
\uparrow & & \downarrow \\
m\ell\mathsf{IGL} \vdash \varphi & \xleftarrow{\Sigma^1_1\text{-Det}} & \mathcal{P}\mathsf{IGL} \models \varphi
\end{array}$$

IGL

We prove completeness with less determinacy:



SOME RULES OF $\mathsf{c\ell IGL}$ — I

$$\mathsf{id}\ \frac{}{\mathbf{R}, \Gamma, x : P \vdash \Delta, x : P}$$

$$\mathsf{tr}\ \frac{\mathbf{R}, xRy, yRz, xRz, \Gamma \vdash \Delta}{\mathbf{R}, xRy, yRz, \Gamma \vdash \Delta}$$

$$\wedge\mathrm{l}\ \frac{\mathbf{R}, \Gamma, x : A \wedge B, x : A, x : B \vdash \Delta}{\mathbf{R}, \Gamma, x : A \wedge B \vdash \Delta}$$

$$\to\mathrm{l}\ \frac{\mathbf{R}, \Gamma, x : A \to B \vdash \Delta, x : A \qquad \mathbf{R}, \Gamma, x : A \to B, x : B \vdash \Delta}{\mathbf{R}, \Gamma, x : A \to B \vdash \Delta}$$

SOME RULES OF cℓIGL — II

$$\Diamond\mathrm{r}\ \frac{\mathbf{R}, \Gamma \vdash \Delta, x : \Diamond A, \{y : A \mid xRy\}}{\mathbf{R}, \Gamma \vdash \Delta, x : \Diamond A}$$

$$\Box\mathrm{l}\ \frac{\mathbf{R}, \Gamma, x : \Box A, \{y : A \mid xRy\} \vdash \Delta}{\mathbf{R}, \Gamma, x : \Box A \vdash \Delta}$$

$$\Diamond\mathrm{l}\ \frac{\mathbf{R}, xRy, \Gamma, x : \Diamond A, y : A \vdash \Delta}{\mathbf{R}, \Gamma, x : \Diamond A \vdash \Delta}\ (y \text{ is fresh})$$

Some rules of $\mathsf{c\ell IGL}$ — III

Non-invertible rules:

$$\to\text{r}\ \frac{\mathbf{R}, \Gamma, x : A \vdash x : B}{\mathbf{R}, \Gamma \vdash \Delta, x : A \to B}$$

$$\Box\text{r}\ \frac{\mathbf{R}, xRy, \Gamma \vdash y : A}{\mathbf{R}, \Gamma \vdash \Delta, x : \Box A}\ (y \text{ is fresh})$$

LOOPS

Loop v_S from $\mathbf{R}, \Gamma \vdash \Delta$ to $\mathbf{R}', \Gamma' \vdash \Delta'$:

- if $x : \varphi \in \Gamma'$ then $v_S(x) : \varphi \in \Gamma$;
- if $x : \varphi \in \Delta'$ then $v_S(x) : \varphi \in \Delta$;
- if $xR'y$ then $v_S(x) R v_S(y)$;
- if there is $x \in \mathrm{Var}(R')$ such that $x R v_S(x) \in \mathbf{R}$.

IGL PROVES $\Box(\Box P \to P) \to \Box P$

$$
\dfrac{
  \text{id}\ \dfrac{}{xRy, x : \Box(\Box P \to P), y : \Box P \to P, y : P \vdash y : P}
  \qquad
  \Box\text{l}\ \dfrac{\text{tr}\ \dfrac{xRy, yRz, xRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P \quad (=: S)}{xRy, yRz, x : \Box(\Box P \to P), y : \Box P \to P \vdash z : P}}{xRy, x : \Box(\Box P \to P), y : \Box P \to P \vdash y : P, y : \Box P}
}{xRy, x : \Box(\Box P \to P), y : \Box P \to P \vdash y : P}\ \to\text{l}
$$

$$
\to\text{r}\ \dfrac{\Box\text{r}\ \dfrac{\Box\text{l}\ \dfrac{xRy, x : \Box(\Box P \to P), y : \Box P \to P \vdash y : P}{xRy, x : \Box(\Box P \to P) \vdash y : P (=: S')}}{x : \Box(\Box P \to P) \vdash x : \Box P}}{\vdash x : \Box(\Box P \to P) \to \Box P}
$$

$$l(S) = S' \text{ and } v_S(x) = x, v_S(y) = z.$$

Predicate Kripke Frames

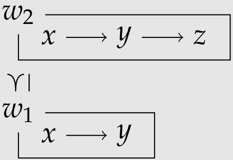
Tuple $M = \langle W, \preceq, \{D_w\}_{w\in W}, \{Pr_w\}_{w\in W}, \{R_w\}_{w\in W}\rangle$ such that:

1. W is a non-empty set of possible worlds;
2. the intuitionistic relation $\preceq$ is a partial order on W;
3. $\{D_w\}_{w\in W}$ is a family of domains $D_w \subseteq \text{Var}$;
4. $\{Pr_w\}_{w\in W}$ is a family of mappings $Pr_w : \text{Prop} \to \mathcal{P}(D_w)$;
5. $\{R_w\}_{w\in W}$ is a family of modal relations $R_w \subseteq D_w \times D_w$;
6. all relations are monotone in $\preceq$, i.e., if $w \preceq w'$, then we have $D_w \subseteq D_{w'}$, $Pr_w \subseteq Pr_{w'}$, and $R_w \subseteq R_{w'}$.

If $M = \langle W, \preceq, \{D_w\}_{w \in W}, \{Pr_w\}_{w \in W}, \{R_w\}_{w \in W} \rangle$, then

- $M, w \models x : P$ iff $x \in Pr_w(P)$.
- $M, w \not\models x : \bot$.
- $M, w \models x : A \wedge B$ iff $M, w \models x : A$ and $M, w \models x : B$.
- $M, w \models x : A \vee B$ iff $M, w \models x : A$ or $M, w \models x : B$.
- $M, w \models x : A \rightarrow B$ iff for all $w' \succeq w$, if $M, w' \models x : A$ then $M, w' \models x : B$.
- $M, w \models x : \Box A$ iff, for all $w' \succeq w$ and for all $y \in D_{w'}$, if $xR_{w'}y$ then $M, w' \models y : A$.
- $M, w \models x : \Diamond A$ iff there is $y \in D_w$ such that xR_wy and $M, w \models y : A$.

IGL does not prove $\Diamond P \to \Diamond(P \wedge \neg\Diamond P)$



IGL does not prove $\Diamond P \to \Diamond(P \wedge \neg\Diamond P)$

$$
\mathsf{\to r}\,\dfrac{\Diamond\mathrm{l}\,\dfrac{\Diamond\mathrm{r}\,\dfrac{\wedge\mathrm{r}\,\dfrac{\mathsf{id}\,\dfrac{}{xRy, x : \Diamond P, y : P \vdash x : \Diamond(P \wedge \neg\Diamond P), y : P} \qquad \to\mathrm{r}\,\dfrac{\Diamond\mathrm{l}\,\dfrac{\mathsf{tr}\,\dfrac{xRy, yRz, xRz, x : \Diamond P, y : \Diamond P, y : P, z : P \vdash y : \bot\ (*)}{xRy, yRz, x : \Diamond P, y : \Diamond P, y : P, z : P \vdash y : \bot}}{xRy, x : \Diamond P, y : \Diamond P, y : P \vdash y : \bot}}{xRy, x : \Diamond P, y : P \vdash x : \Diamond(P \wedge \neg\Diamond P), y : \neg\Diamond P\ (*)}}{xRy, x : \Diamond P, y : P \vdash x : \Diamond(P \wedge \neg\Diamond P), y : P \wedge \neg\Diamond P}}{xRy, x : \Diamond P, y : P \vdash x : \Diamond(P \wedge \neg\Diamond P)}}{x : \Diamond P \vdash x : \Diamond(P \wedge \neg\Diamond P)}}{\vdash x : \Diamond P \to \Diamond(P \wedge \neg\Diamond P)}
$$

Results

Theorem

ATR_0 *proves the Kripke completeness of* IGL.

Theorem

IGL *is recursively enumerable.*

Open Questions

Question

Is IGL *recursive?*

Question

Does IGL *have the finite model property?*

Question

Does IGL *have a finite Hilbert-style axiomatization?*

REFERENCES

[1] Aguilera, Pacheco, "IGL without sharps", preprint soon™.

[2] Das, van der Giessen, Marin, "Intuitionistic Gödel-Löb logic, à la Simpson: labelled systems and birelational semantics", 2024.

[3] Van der Giessen, "Uniform Interpolation and Admissible Rules: Proof-theoretic investigations into (intuitionistic) modal logics", 2022.