Categoricity-like notions for first-order theories

Mateusz Łełyk

Faculty of Philosophy, University of Warsaw

Ghent, Wormshop
01.09.2024

Day 1: Categoricity of PA

Motivations

Dedekind Categoricity Theorem (1888). *There is a sentence σ in second order logic of the form $\forall X \varphi(X)$, where $\varphi(X)$ only has first order quantifiers, such that σ holds in a structure $\mathcal{M}$ iff $\mathcal{M} \cong (\mathbb{N}, S, 0)$, where S is the successor function.*

Zermelo Quasi-categoricity Theorem (1930). *There is a sentence θ in second order logic of the form $\forall X \psi(X)$, where $\psi(X)$ only has first order quantifiers, such that θ holds in a structure $\mathcal{M}$ iff $\mathcal{M} \cong (V_\kappa, \in)$, where κ is a strongly inaccessible cardinal.*

Question

Are "first-order counterparts" of these second order systems in a sense categorical?

Background definitions: interpretability

Let U, V be any first-order theories in languages $\mathcal{L}_U$ and $\mathcal{L}_V$, respectively.

Background definitions: interpretability

Let U, V be any first-order theories in languages $\mathcal{L}_U$ and $\mathcal{L}_V$, respectively. We say that

$$\sigma : \mathrm{Form}_{\mathcal{L}_U} \rightarrow \mathrm{Form}_{\mathcal{L}_V}$$

is a *translation function* iff it satisies:

Background definitions: interpretability

Let U, V be any first-order theories in languages $\mathcal{L}_U$ and $\mathcal{L}_V$, respectively. We say that

$$\sigma : \mathrm{Form}_{\mathcal{L}_U} \to \mathrm{Form}_{\mathcal{L}_V}$$

is a *translation function* iff it satisies:

1. there is a designated *domain formula* $\mathcal{L}_V$-formula $\delta(x)$.

Background definitions: interpretability

Let U, V be any first-order theories in languages $\mathcal{L}_U$ and $\mathcal{L}_V$, respectively. We say that

$$\sigma : \mathrm{Form}_{\mathcal{L}_U} \to \mathrm{Form}_{\mathcal{L}_V}$$

is a *translation function* iff it satisies:

1. there is a designated *domain formula* $\mathcal{L}_V$-formula $\delta(x)$.
2. there is a designated mapping $P \mapsto F_P$ that translates each n-ary $\mathcal{L}_U$-predicate P into some n-ary $\mathcal{L}_V$-formula F_P (including the case when P is the equality relation).

Background definitions: interpretability

Let U, V be any first-order theories in languages $\mathcal{L}_U$ and $\mathcal{L}_V$, respectively. We say that

$$\sigma : \mathrm{Form}_{\mathcal{L}_U} \to \mathrm{Form}_{\mathcal{L}_V}$$

is a *translation function* iff it satisies:

1. there is a designated *domain formula* $\mathcal{L}_V$-formula $\delta(x)$.
2. there is a designated mapping $P \mapsto F_P$ that translates each n-ary $\mathcal{L}_U$-predicate P into some n-ary $\mathcal{L}_V$-formula F_P (including the case when P is the equality relation).
3. σ commutes with propositional connectives, and is subject to:

$$\sigma(\forall x \varphi) = \forall x\, (\delta(x) \to \sigma(\varphi)).$$

Background definitions: interpretability

We say that $\mathcal{I}$ *is an interpretation of* U *in* V, written $U \trianglelefteq^{\mathcal{I}} V$, if $\mathcal{I}$ specifies a *translation function*

$$\sigma : \mathrm{Form}_{\mathcal{L}_U} \to \mathrm{Form}_{\mathcal{L}_V}$$

such that for each $\varphi \in \mathcal{L}_U$,

$$U \vdash \varphi \Rightarrow V \vdash \sigma(\varphi).$$

Background definitions: interpretability

We say that $\mathcal{I}$ *is an interpretation of* U *in* V, written $U \trianglelefteq^{\mathcal{I}} V$, if $\mathcal{I}$ specifies a *translation function*

$$\sigma : \mathrm{Form}_{\mathcal{L}_U} \to \mathrm{Form}_{\mathcal{L}_V}$$

such that for each $\varphi \in \mathcal{L}_U$,

$$U \vdash \varphi \Rightarrow V \vdash \sigma(\varphi).$$

Remark

The above definition is not ultimately general.

Background definitions: interpretability

We say that $\mathcal{I}$ *is an interpretation of* U *in* V, written $U \trianglelefteq^{\mathcal{I}} V$, if $\mathcal{I}$ specifies a *translation function*

$$\sigma : \mathrm{Form}_{\mathcal{L}_U} \to \mathrm{Form}_{\mathcal{L}_V}$$

such that for each $\varphi \in \mathcal{L}_U$,

$$U \vdash \varphi \Rightarrow V \vdash \sigma(\varphi).$$

Remark

The above definition is not ultimately general. Additionally one can allow

- *U-objects to be coded by tuples of V-objects;*
- *to use parameters.*

Interpretability from a model-theoretical perspective

Remark

Each translation $\sigma : \mathrm{Form}_{\mathcal{L}_U} \to \mathrm{Form}_{\mathcal{L}_V}$ *gives rise to a uniform transformation of* $\mathcal{L}_V$*-structures into* $\mathcal{L}_U$*-structures.*

Interpretability from a model-theoretical perspective

Remark

Each translation $\sigma : \mathrm{Form}_{\mathcal{L}_U} \to \mathrm{Form}_{\mathcal{L}_V}$ *gives rise to a uniform transformation of* $\mathcal{L}_V$*-structures into* $\mathcal{L}_U$*-structures.*

That is, given any $\mathcal{L}_V$-structure $\mathcal{M}$ we obtain an $\mathcal{L}_U$-structure $\sigma(\mathcal{M})$ such that

Interpretability from a model-theoretical perspective

Remark

Each translation $\sigma : \mathrm{Form}_{\mathcal{L}_U} \to \mathrm{Form}_{\mathcal{L}_V}$ *gives rise to a uniform transformation of* $\mathcal{L}_V$*-structures into* $\mathcal{L}_U$*-structures.*

That is, given any $\mathcal{L}_V$-structure $\mathcal{M}$ we obtain an $\mathcal{L}_U$-structure $\sigma(\mathcal{M})$ such that

1. domain is the set defined in $\mathcal{M}$ by δ;

Interpretability from a model-theoretical perspective

Remark

Each translation $\sigma : \mathrm{Form}_{\mathcal{L}_U} \to \mathrm{Form}_{\mathcal{L}_V}$ *gives rise to a uniform transformation of* $\mathcal{L}_V$*-structures into* $\mathcal{L}_U$*-structures.*

That is, given any $\mathcal{L}_V$-structure $\mathcal{M}$ we obtain an $\mathcal{L}_U$-structure $\sigma(\mathcal{M})$ such that

1. domain is the set defined in $\mathcal{M}$ by δ;
2. each predicate P (including the equality predicate) is interpreted as the set defined in $\mathcal{M}$ by F_P.

Interpretability from a model-theoretical perspective

Remark

Each translation $\sigma : \mathrm{Form}_{\mathcal{L}_U} \to \mathrm{Form}_{\mathcal{L}_V}$ *gives rise to a uniform transformation of* $\mathcal{L}_V$*-structures into* $\mathcal{L}_U$*-structures.*

That is, given any $\mathcal{L}_V$-structure $\mathcal{M}$ we obtain an $\mathcal{L}_U$-structure $\sigma(\mathcal{M})$ such that

1. domain is the set defined in $\mathcal{M}$ by δ;
2. each predicate P (including the equality predicate) is interpreted as the set defined in $\mathcal{M}$ by F_P.

Moreover, an interpretation $\mathcal{I}$ based on σ such that $U \trianglelefteq^{\mathcal{I}} V$ gives rise to an *internal model construction that* ***uniformly*** *builds a model* $\mathcal{M}^{\mathcal{I}} \models U$ *for any* $\mathcal{M} \models V$, where $\mathcal{M}^{\mathcal{I}} := \sigma^{\mathcal{M}}$.

Interpreting structures

Definition

A model $\mathcal{N}$ for a language $\mathcal{L}_1$ is interpretable in a model $\mathcal{M}$ for a language $\mathcal{L}_2$ iff there is a translation $\sigma : \mathcal{L}_1 \to \mathcal{L}_2$ such that $\mathcal{N} = \sigma^{\mathcal{M}}$.

Remark

Translations and interpretations can be composed.

Interpreting structures

Definition

A model $\mathcal{N}$ for a language $\mathcal{L}_1$ is interpretable in a model $\mathcal{M}$ for a language $\mathcal{L}_2$ iff there is a translation $\sigma : \mathcal{L}_1 \to \mathcal{L}_2$ such that $\mathcal{N} = \sigma^{\mathcal{M}}$.

Remark

Translations and interpretations can be composed. Given $T \trianglelefteq^{\mathcal{J}} U \trianglelefteq^{\mathcal{I}} V$, to define $\mathcal{I} \circ \mathcal{J}$, just compute the T model given by $\mathcal{J}$ in the U-model given by $\mathcal{I}$.

Some famous interpretabilities

- $\mathsf{Th}(\mathbb{Z}, +, \times) \trianglelefteq \mathsf{Th}(\mathbb{N}, +, \times)$.

Some famous interpretabilities

- $\mathrm{Th}(\mathbb{Z}, +, \times) \trianglelefteq \mathrm{Th}(\mathbb{N}, +, \times)$. Integer numbers coded as pairs of natural numbers (elements coded by pairs).
- $\mathrm{ZF} - \mathrm{Infinity} + \neg\mathrm{Infinity} =: \mathrm{ZF}_{fin} \trianglelefteq \mathrm{PA}$.

Some famous interpretabilities

- $\mathrm{Th}(\mathbb{Z}, +, \times) \trianglelefteq \mathrm{Th}(\mathbb{N}, +, \times)$. Integer numbers coded as pairs of natural numbers (elements coded by pairs).
- $\mathsf{ZF} - \mathsf{Infinity} + \neg\mathsf{Infinity} =: \mathsf{ZF}_{fin} \trianglelefteq \mathsf{PA}$. Sets coded as binary strings.

Some famous interpretabilities

- $\mathrm{Th}(\mathbb{Z}, +, \times) \trianglelefteq \mathrm{Th}(\mathbb{N}, +, \times)$. Integer numbers coded as pairs of natural numbers (elements coded by pairs).
- $\mathsf{ZF} - \mathsf{Infinity} + \neg\mathsf{Infinity} =: \mathsf{ZF}_{fin} \trianglelefteq \mathsf{PA}$. Sets coded as binary strings.
- $\mathsf{PA} \trianglelefteq \mathsf{ZF}_{fin}$.

Some famous interpretabilities

- $\mathrm{Th}(\mathbb{Z}, +, \times) \trianglelefteq \mathrm{Th}(\mathbb{N}, +, \times)$. Integer numbers coded as pairs of natural numbers (elements coded by pairs).
- $\mathsf{ZF} - \mathsf{Infinity} + \neg\mathsf{Infinity} =: \mathsf{ZF}_{fin} \trianglelefteq \mathsf{PA}$. Sets coded as binary strings.
- $\mathsf{PA} \trianglelefteq \mathsf{ZF}_{fin}$. Numbers as finite ordinals.

Some famous interpretabilities

- $\mathrm{Th}(\mathbb{Z}, +, \times) \trianglelefteq \mathrm{Th}(\mathbb{N}, +, \times)$. Integer numbers coded as pairs of natural numbers (elements coded by pairs).
- $\mathrm{ZF} - \mathrm{Infinity} + \neg\mathrm{Infinity} =: \mathrm{ZF}_{fin} \trianglelefteq \mathrm{PA}$. Sets coded as binary strings.
- $\mathrm{PA} \trianglelefteq \mathrm{ZF}_{fin}$. Numbers as finite ordinals.
- PA and $\mathrm{PA} + \neg\mathrm{Con}(\mathrm{PA})$ are mutually interpretable.

Some famous interpretabilities

- $\mathrm{Th}(\mathbb{Z}, +, \times) \trianglelefteq \mathrm{Th}(\mathbb{N}, +, \times)$. Integer numbers coded as pairs of natural numbers (elements coded by pairs).
- $\mathsf{ZF} - \mathsf{Infinity} + \neg\mathsf{Infinity} =: \mathsf{ZF}_{fin} \trianglelefteq \mathsf{PA}$. Sets coded as binary strings.
- $\mathsf{PA} \trianglelefteq \mathsf{ZF}_{fin}$. Numbers as finite ordinals.
- PA and $\mathsf{PA} + \neg\mathrm{Con}(\mathsf{PA})$ are mutually interpretable.
- $\mathsf{ZF} - \mathsf{Powerset} + \forall x\big(|x| \leq \aleph_0\big) \trianglelefteq \mathsf{Z}_2 + \Pi^1_\infty - \mathsf{AC}$.

Some famous interpretabilities

- $\mathrm{Th}(\mathbb{Z}, +, \times) \trianglelefteq \mathrm{Th}(\mathbb{N}, +, \times)$. Integer numbers coded as pairs of natural numbers (elements coded by pairs).
- $\mathrm{ZF} - \mathrm{Infinity} + \neg\mathrm{Infinity} =: \mathrm{ZF}_{fin} \trianglelefteq \mathrm{PA}$. Sets coded as binary strings.
- $\mathrm{PA} \trianglelefteq \mathrm{ZF}_{fin}$. Numbers as finite ordinals.
- PA and $\mathrm{PA} + \neg\mathrm{Con}(\mathrm{PA})$ are mutually interpretable.
- $\mathrm{ZF} - \mathrm{Powerset} + \forall x\big(|x| \leq \aleph_0\big) \trianglelefteq \mathrm{Z}_2 + \Pi^1_\infty - \mathrm{AC}$. Sets as well-founded trees (equality redefined!)

Some famous interpretabilities

- $\mathrm{Th}(\mathbb{Z}, +, \times) \trianglelefteq \mathrm{Th}(\mathbb{N}, +, \times)$. Integer numbers coded as pairs of natural numbers (elements coded by pairs).
- $\mathrm{ZF} - \mathrm{Infinity} + \neg\mathrm{Infinity} =: \mathrm{ZF}_{fin} \trianglelefteq \mathrm{PA}$. Sets coded as binary strings.
- $\mathrm{PA} \trianglelefteq \mathrm{ZF}_{fin}$. Numbers as finite ordinals.
- PA and $\mathrm{PA} + \neg\mathrm{Con}(\mathrm{PA})$ are mutually interpretable.
- $\mathrm{ZF} - \mathrm{Powerset} + \forall x\big(|x| \leq \aleph_0\big) \trianglelefteq \mathrm{Z}_2 + \Pi^1_\infty - \mathrm{AC}$. Sets as well-founded trees (equality redefined!)
- $\mathrm{Z}_2 + \Pi^1_\infty - \mathrm{AC} \trianglelefteq \mathrm{ZF} - \mathrm{Powerset} + \forall x\big(|x| \leq \aleph_0\big)$.

Some famous interpretabilities

- $\mathrm{Th}(\mathbb{Z}, +, \times) \trianglelefteq \mathrm{Th}(\mathbb{N}, +, \times)$. Integer numbers coded as pairs of natural numbers (elements coded by pairs).
- $\mathrm{ZF} - \mathrm{Infinity} + \neg\mathrm{Infinity} =: \mathrm{ZF}_{fin} \trianglelefteq \mathrm{PA}$. Sets coded as binary strings.
- $\mathrm{PA} \trianglelefteq \mathrm{ZF}_{fin}$. Numbers as finite ordinals.
- PA and $\mathrm{PA} + \neg\mathrm{Con}(\mathrm{PA})$ are mutually interpretable.
- $\mathrm{ZF} - \mathrm{Powerset} + \forall x\big(|x| \leq \aleph_0\big) \trianglelefteq \mathrm{Z}_2 + \Pi^1_\infty - \mathrm{AC}$. Sets as well-founded trees (equality redefined!)
- $\mathrm{Z}_2 + \Pi^1_\infty - \mathrm{AC} \trianglelefteq \mathrm{ZF} - \mathrm{Powerset} + \forall x\big(|x| \leq \aleph_0\big)$.
- $\mathrm{ZF} + V = L \trianglelefteq \mathrm{ZF}$

Biinterpretability

U is *biinterpretable with* V, $U \simeq V$ iff there are

- interpretations $\mathcal{I}$ and $\mathcal{J}$ with $U \trianglelefteq^{\mathcal{I}} V$, and $V \trianglelefteq^{\mathcal{J}} U$ and

Biinterpretability

U is *biinterpretable with* V, $U \simeq V$ iff there are

- interpretations $\mathcal{I}$ and $\mathcal{J}$ with $U \trianglelefteq^{\mathcal{I}} V$, and $V \trianglelefteq^{\mathcal{J}} U$ and
- a binary U-formula F such that F is, U-verifiably, an isomorphism between id_U (the identity interpretation on U) and $\mathcal{J} \circ \mathcal{I}$.

Biinterpretability

U is *biinterpretable with* V, $U \simeq V$ iff there are

- interpretations $\mathcal{I}$ and $\mathcal{J}$ with $U \trianglelefteq^{\mathcal{I}} V$, and $V \trianglelefteq^{\mathcal{J}} U$ and
- a binary U-formula F such that F is, U-verifiably, an isomorphism between id_U (the identity interpretation on U) and $\mathcal{J} \circ \mathcal{I}$.
- a binary V-formula G such that G is, V-verifiably, an isomorphism between id_V (the identity interpretation on V) and $\mathcal{I} \circ \mathcal{J}$.

Biinterpretability

U is *biinterpretable with* V, $U \simeq V$ iff there are

- interpretations $\mathcal{I}$ and $\mathcal{J}$ with $U \trianglelefteq^{\mathcal{I}} V$, and $V \trianglelefteq^{\mathcal{J}} U$ and
- a binary U-formula F such that F is, U-verifiably, an isomorphism between id_U (the identity interpretation on U) and $\mathcal{J} \circ \mathcal{I}$.
- a binary V-formula G such that G is, V-verifiably, an isomorphism between id_V (the identity interpretation on V) and $\mathcal{I} \circ \mathcal{J}$.

Model theoretic picture.

Some examples

- Let $U := \forall x P(x)$, $V := \forall x \neg P(x)$. Then U and V are biinterpretable. ("The stupid example").

Some examples

- Let $U := \forall x P(x)$, $V := \forall x \neg P(x)$. Then U and V are biinterpretable. ("The stupid example").
- PA is biinterpretable with ZF_{fin} + Every set is contained in a transitive set.

Some examples

- Let $U := \forall x P(x)$, $V := \forall x \neg P(x)$. Then U and V are biinterpretable. ("The stupid example").
- PA is biinterpretable with ZF_{fin} + Every set is contained in a transitive set.
- CT is biinterpretable with ACA + "Every set is arithmetically definable".

Some examples

- Let $U := \forall x P(x)$, $V := \forall x \neg P(x)$. Then U and V are biinterpretable. ("The stupid example").
- PA is biinterpretable with ZF_{fin} + Every set is contained in a transitive set.
- CT is biinterpretable with ACA + "Every set is arithmetically definable".
- $\mathrm{Z}_2 + \Pi^1_\infty - \mathrm{AC}$ and $\mathrm{ZF} - \mathrm{Powerset} + \forall x\big(|x| \leq \aleph_0\big)$ are biinterpretable.

Some examples

- + Let $U := \forall x P(x)$, $V := \forall x \neg P(x)$. Then U and V are biinterpretable. ("The stupid example").
- + PA is biinterpretable with ZF_{fin} + Every set is contained in a transitive set.
- + CT is biinterpretable with ACA + "Every set is arithmetically definable".
- + $\mathsf{Z}_2 + \Pi^1_\infty - \mathsf{AC}$ and $\mathsf{ZF} - \mathsf{Powerset} + \forall x \big(|x| \leq \aleph_0\big)$ are biinterpretable.
- – CT is **not** biinterpretable with ACA.

Some examples

+ Let $U := \forall x P(x)$, $V := \forall x \neg P(x)$. Then U and V are biinterpretable. ("The stupid example").
+ PA is biinterpretable with ZF_{fin} + Every set is contained in a transitive set.
+ CT is biinterpretable with ACA + "Every set is arithmetically definable".
+ $\mathrm{Z}_2 + \Pi^1_\infty - \mathrm{AC}$ and $\mathrm{ZF} - \mathrm{Powerset} + \forall x\big(|x| \leq \aleph_0\big)$ are biinterpretable.

– CT is **not** biinterpretable with ACA.
– PA is **not** biinterpretable with ZF_{fin}.

Some examples

- \+ Let $U := \forall x P(x)$, $V := \forall x \neg P(x)$. Then U and V are biinterpretable. ("The stupid example").
- \+ PA is biinterpretable with ZF_{fin} + Every set is contained in a transitive set.
- \+ CT is biinterpretable with ACA + "Every set is arithmetically definable".
- \+ $\mathrm{Z}_2 + \Pi^1_\infty - \mathrm{AC}$ and $\mathrm{ZF} - \mathrm{Powerset} + \forall x\big(|x| \leq \aleph_0\big)$ are biinterpretable.
- – CT is **not** biinterpretable with ACA.
- – PA is **not** biinterpretable with ZF_{fin}.
- – ZF is **not** biinterpretable with ZF + V= L.

Why biinterpretability?

- If U and V are sequential theories which are biinterpretable through $=$-preserving interpretations, then they have a common definitional extension (Friedmann-Visser Theorem).

Why biinterpretability?

- If U and V are sequential theories which are biinterpretable through $=$-preserving interpretations, then they have a common definitional extension (Friedmann-Visser Theorem).
- If U and V are biinterpretable and one of them is finite, then both are finite.

Why biinterpretability?

- If U and V are sequential theories which are biinterpretable through $=$-preserving interpretations, then they have a common definitional extension (Friedmann-Visser Theorem).
- If U and V are biinterpretable and one of them is finite, then both are finite.
- If $\mathcal{M}$ and $\mathcal{N}$ are biinterpretable, their automorphism groups are isomorphic.

Day 1: Categoricity of PA

Tightness

Tight theories

Definition

A theory U is tight iff for all V_1, V_2 – (deductively closed) extensions of U in $\mathcal{L}_U$

$$V_1 \simeq V_2 \Longrightarrow V_1 = V_2.$$

Tight theories

Definition

A theory U is tight iff for all V_1, V_2 – (deductively closed) extensions of U in $\mathcal{L}_U$

$$V_1 \simeq V_2 \Longrightarrow V_1 = V_2.$$

- Trivially, every complete theory is tight.

Tight theories

Definition

A theory U is tight iff for all V_1, V_2 – (deductively closed) extensions of U in $\mathcal{L}_U$

$$V_1 \simeq V_2 \Longrightarrow V_1 = V_2.$$

+ Trivially, every complete theory is tight.
– PA(P) is not tight.

Tight theories

Definition

A theory U is tight iff for all V_1, V_2 – (deductively closed) extensions of U in $\mathcal{L}_U$

$$V_1 \simeq V_2 \Longrightarrow V_1 = V_2.$$

+ Trivially, every complete theory is tight.
– PA(P) is not tight. Recall the stupid example.

Tight theories

Definition

A theory U is tight iff for all V_1, V_2 – (deductively closed) extensions of U in $\mathcal{L}_U$

$$V_1 \simeq V_2 \Longrightarrow V_1 = V_2.$$

+ Trivially, every complete theory is tight.
– $\mathsf{PA}(P)$ is not tight. Recall the stupid example.

Definition

A theory U is minimalist iff for every $\mathcal{M} \models U$ and every $\mathcal{M} \trianglelefteq \mathcal{N} \models U$ there is **exactly one** $\mathcal{M}$**-definable** embedding $\mathcal{M} \hookrightarrow \mathcal{N}$.

Tight theories

Definition

A theory U is tight iff for all V_1, V_2 – (deductively closed) extensions of U in $\mathcal{L}_U$

$$V_1 \simeq V_2 \Longrightarrow V_1 = V_2.$$

+ Trivially, every complete theory is tight.
– PA(P) is not tight. Recall the stupid example.

Definition

A theory U is minimalist iff for every $\mathcal{M} \models U$ and every $\mathcal{M} \trianglelefteq \mathcal{N} \models U$ there is **exactly one** $\mathcal{M}$**-definable** embedding $\mathcal{M} \hookrightarrow \mathcal{N}$.

Proposition

Every minimalist theory is tight.

Tightness of PA

Theorem (Visser)

PA *is tight.*

Tightness of PA

Theorem (Visser)

PA *is tight.*

Proof.

Show that PA is minimalist.

Tightness of PA

Theorem (Visser)

PA *is tight.*

Proof.

Show that PA is minimalist. Given an $\mathcal{M}$ and $\mathcal{N} \models$ PA such that $\mathcal{M} \trianglelefteq \mathcal{N}$, show that

$$\mathcal{M} \models \text{"For every } x \text{ there is the } x\text{-th } \mathcal{N}\text{-successor of } 0_{\mathcal{N}}\text{."}$$

Tightness of PA

Theorem (Visser)

PA *is tight.*

Proof.

Show that PA is minimalist. Given an $\mathcal{M}$ and $\mathcal{N} \models$ PA such that $\mathcal{M} \trianglelefteq \mathcal{N}$, show that

$$\mathcal{M} \models \text{"For every } x \text{ there is the } x\text{-th } \mathcal{N}\text{-successor of } 0_{\mathcal{N}}.\text{"}$$

This gives rise to the unique **definable** embedding of $\mathcal{M}$ into $\mathcal{N}$. □

Restricted fragments of PA

Let PA_n denote the set of Σ_n-consequences of PA.

Restricted fragments of PA

Let PA_n denote the set of Σ_n-consequences of PA.

Theorem (Enayat)

*For every n, PA_n is **not** tight.*

Restricted fragments of PA

Let PA_n denote the set of Σ_n-consequences of PA.

Theorem (Enayat)

*For every n, PA_n is **not** tight.*

Recall the canonical models in which PA *fails*:

Restricted fragments of PA

Let PA_n denote the set of Σ_n-consequences of PA.

Theorem (Enayat)

*For every n, PA_n is **not** tight.*

Recall the canonical models in which PA *fails*: suppose $\mathcal{M} \models \mathrm{PA}$ and set

$$K^n(\mathcal{M}) := \{a \in M : \exists \phi(x) \in \Sigma_n \;\; \mathcal{M} \models \phi(a) \wedge \exists ! x \phi(x)\}.$$

It's fairly easy to check that

Restricted fragments of PA

Let PA_n denote the set of Σ_n-consequences of PA.

Theorem (Enayat)

*For every n, PA_n is **not** tight.*

Recall the canonical models in which PA *fails*: suppose $\mathcal{M} \models \mathrm{PA}$ and set

$$K^n(\mathcal{M}) := \{a \in M : \exists \phi(x) \in \Sigma_n \ \ \mathcal{M} \models \phi(a) \wedge \exists! x \phi(x)\}.$$

It's fairly easy to check that

- $K^n(\mathcal{M}) \subseteq \mathcal{M}$

Restricted fragments of PA

Let PA_n denote the set of Σ_n-consequences of PA.

Theorem (Enayat)

For every n, PA_n is ***not*** *tight.*

Recall the canonical models in which PA *fails*: suppose $\mathcal{M} \models \mathrm{PA}$ and set

$$K^n(\mathcal{M}) := \{a \in M : \exists \phi(x) \in \Sigma_n \ \ \mathcal{M} \models \phi(a) \wedge \exists! x \phi(x)\}.$$

It's fairly easy to check that

- $K^n(\mathcal{M}) \subseteq \mathcal{M}$ and moreover

Restricted fragments of PA

Let PA_n denote the set of Σ_n-consequences of PA.

Theorem (Enayat)

*For every n, PA_n is **not** tight.*

Recall the canonical models in which PA *fails*: suppose $\mathcal{M} \models \mathrm{PA}$ and set

$$K^n(\mathcal{M}) := \{a \in M : \exists \phi(x) \in \Sigma_n \ \ \mathcal{M} \models \phi(a) \wedge \exists! x \phi(x)\}.$$

It's fairly easy to check that

- $K^n(\mathcal{M}) \subseteq \mathcal{M}$ and moreover
- (TLDR: $K^n(\mathcal{M}) \preceq_{\Sigma_n} \mathcal{M}$)

Restricted fragments of PA

Let PA_n denote the set of Σ_n-consequences of PA.

Theorem (Enayat)

*For every n, PA_n is **not** tight.*

Recall the canonical models in which PA *fails*: suppose $\mathcal{M} \models \mathrm{PA}$ and set

$$K^n(\mathcal{M}) := \{a \in M : \exists \phi(x) \in \Sigma_n \ \ \mathcal{M} \models \phi(a) \wedge \exists! x \phi(x)\}.$$

It's fairly easy to check that

- $K^n(\mathcal{M}) \subseteq \mathcal{M}$ and moreover
- (TLDR: $K^n(\mathcal{M}) \preceq_{\Sigma_n} \mathcal{M}$) for every $\psi(x) \in \Sigma_n$ and every $a \in K^n(\mathcal{M})$

$$\mathcal{M} \models \psi(a) \Rightarrow K^n(\mathcal{M}) \models \psi(a).$$

Restricted fragments of PA

Let PA_n denote the set of Σ_n-consequences of PA.

Theorem (Enayat)

*For every n, PA_n is **not** tight.*

Recall the canonical models in which PA *fails*: suppose $\mathcal{M} \models \mathrm{PA}$ and set

$$K^n(\mathcal{M}) := \{a \in M : \exists \phi(x) \in \Sigma_n \ \ \mathcal{M} \models \phi(a) \wedge \exists! x \phi(x)\}.$$

It's fairly easy to check that

- $K^n(\mathcal{M}) \subseteq \mathcal{M}$ and moreover
- (TLDR: $K^n(\mathcal{M}) \preceq_{\Sigma_n} \mathcal{M}$) for every $\psi(x) \in \Sigma_n$ and every $a \in K^n(\mathcal{M})$

$$\mathcal{M} \models \psi(a) \Rightarrow K^n(\mathcal{M}) \models \psi(a).$$

In particular $K^n(\mathcal{M}) \models \mathrm{PA}_n$.

$\mathcal{I} : \mathbb{N} \trianglelefteq K^n(\mathcal{M})$

Recall that there are arithmetical formulae $\mathsf{Sat}_n(x, y)$ such that for each Σ_n formula $\phi(x)$

$$\mathsf{I}\Sigma_1 \vdash \forall y\big(\mathsf{Sat}_n(\ulcorner\phi(x)\urcorner, y) \equiv \phi(y)\big).$$

$\mathcal{I} : \mathbb{N} \trianglelefteq K^n(\mathcal{M})$

Recall that there are arithmetical formulae $\mathrm{Sat}_n(x, y)$ such that for each Σ_n formula $\phi(x)$

$$\mathrm{I}\Sigma_1 \vdash \forall y \big(\mathrm{Sat}_n(\ulcorner \phi(x) \urcorner, y) \equiv \phi(y)\big).$$

Observe that for every $x \in K^n(\mathcal{M})$

$$K^n(\mathcal{M}) \models \text{"}x \text{ is below the least } \Sigma_n \text{ definition of something"} \iff x \in \mathbb{N}.$$

$\mathcal{I} : \mathbb{N} \trianglelefteq K^n(\mathcal{M})$

Recall that there are arithmetical formulae $\mathrm{Sat}_n(x, y)$ such that for each Σ_n formula $\phi(x)$

$$\mathrm{I}\Sigma_1 \vdash \forall y \big(\mathrm{Sat}_n(\ulcorner \phi(x) \urcorner, y) \equiv \phi(y)\big).$$

Observe that for every $x \in K^n(\mathcal{M})$

$$K^n(\mathcal{M}) \models \text{"}x \text{ is below the least } \Sigma_n \text{ definition of something"} \iff x \in \mathbb{N}.$$

Hence $\mathbb{N} \trianglelefteq K^n(\mathcal{M})$.

$\mathcal{J} : K^n(\mathcal{M}) \trianglelefteq \mathbb{N}$

1. Choose $\mathcal{M}$ to be a model of PA obtained from the Arithmetized Completeness Theorem.

$\mathcal{J} : K^n(\mathcal{M}) \trianglelefteq \mathbb{N}$

1. Choose $\mathcal{M}$ to be a model of PA obtained from the Arithmetized Completeness Theorem.
2. Observe that not only $\mathcal{M}$, but also a satisfaction predicate for $\mathcal{M}$ is arithmetically definable.

$\mathcal{J} : K^n(\mathcal{M}) \trianglelefteq \mathbb{N}$

1. Choose $\mathcal{M}$ to be a model of PA obtained from the Arithmetized Completeness Theorem.
2. Observe that not only $\mathcal{M}$, but also a satisfaction predicate for $\mathcal{M}$ is arithmetically definable.
3. Copy the definition of $K^n(\mathcal{M})$.

$K^n(\mathcal{M}) \simeq \mathbb{N}$

- $\mathcal{J} \circ \mathcal{I} \sim \mathrm{id}_{\mathbb{N}}$: map n to the n-th element of $\mathcal{I} \circ \mathcal{J}$.

$K^n(\mathcal{M}) \simeq \mathbb{N}$

- $\mathcal{J} \circ \mathcal{I} \sim \mathrm{id}_{\mathbb{N}}$: map n to the n-th element of $\mathcal{I} \circ \mathcal{J}$.
- $\mathcal{I} \circ \mathcal{J} \sim \mathrm{id}_{K^n(\mathcal{M})}$:

$K^n(\mathcal{M}) \simeq \mathbb{N}$

- $\mathcal{J} \circ \mathcal{I} \sim \mathrm{id}_{\mathbb{N}}$: map n to the n-th element of $\mathcal{I} \circ \mathcal{J}$.
- $\mathcal{I} \circ \mathcal{J} \sim \mathrm{id}_{K^n(\mathcal{M})}$:
  1. given $x \in K^n(\mathcal{M})$ find its least Σ_n-definition $\phi_x \in \mathbb{N}$.

$K^n(\mathcal{M}) \simeq \mathbb{N}$

- $\mathcal{J} \circ \mathcal{I} \sim \mathrm{id}_{\mathbb{N}}$: map n to the n-th element of $\mathcal{I} \circ \mathcal{J}$.
- $\mathcal{I} \circ \mathcal{J} \sim \mathrm{id}_{K^n(\mathcal{M})}$:
  1. given $x \in K^n(\mathcal{M})$ find its least Σ_n-definition $\phi_x \in \mathbb{N}$.
  2. map x to whatever ϕ_x defines (according to $\mathbb{N}$-definable satisfaction predicate) in $\mathcal{I} \circ \mathcal{J}$.

Day 1: Categoricity of PA

Solid theories

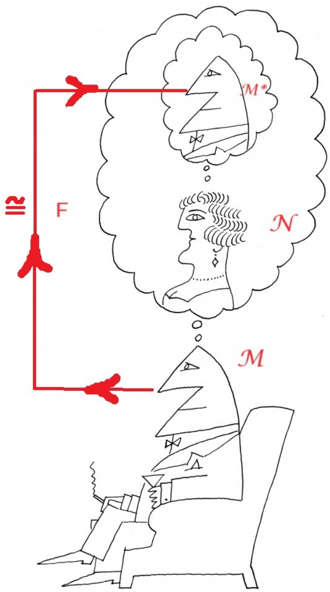
Retracts

$\mathcal{M}$ is a *retract* of $\mathcal{N}$ iff there are

- interpretations $\mathcal{I}$ and $\mathcal{J}$ with $\mathcal{M} \trianglelefteq^{\mathcal{I}} \mathcal{N}$, and $\mathcal{N} \trianglelefteq^{\mathcal{J}} \mathcal{M}^*$
- a binary $\mathcal{M}$-formula F such that F is, $\mathcal{M}$-verifiably, an isomorphism between $\mathrm{id}_{\mathcal{M}}$ (the identity interpretation on $\mathcal{M}$) and $\mathcal{J} \circ \mathcal{I}$.

Example

$\mathbb{N}$ is a retract of $(\mathbb{Z}[X]_{\geq 0}, +, \times)$.



Based on a picture of Saul Steinberg (1962).

Solidity

Definition

A theory U is solid iff whenever $\mathcal{M}, \mathcal{N} \models U$ and

$$\mathcal{I} : \mathcal{M} \trianglelefteq \mathcal{N}$$
$$\mathcal{J} : \mathcal{N} \trianglelefteq \mathcal{N}^{\mathcal{J}}$$

witness that $\mathcal{M}$ is a retract of $\mathcal{N}$, then there is an $\mathcal{N}$-definable isomorphism $\mathcal{N} \sim \mathcal{N}^{\mathcal{J}}$.

Remark

Minimalist $\Longrightarrow$ *Solidity* $\Longrightarrow$ *Tightness.*

Corollary

PA *is solid but for every* n, PA_n *is not solid.*