Towards Intuitionistic Polymodal Provability Logic

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Classical Provability Logic

- Language: $\wedge$, $\vee$, $\rightarrow$, $\bot$, $\Box$.
- $(\Box A)^*$ interpreted as "A^* is provable in PA".
- Solovay: $\mathsf{GL} = \{A : \forall * \ (\mathsf{PA} \vdash A^*)\}$.
- All sound extensions of PA have the same provability logic.
- $\mathsf{GL} :=$
  - Axioms of Classical Logic.
  - $\mathsf{K} := \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$.
  - $\mathsf{Lob} := \Box(\Box A \rightarrow A) \rightarrow \Box A$.

Intuitionistic Provability Logic

- $(\Box A)^*$ interpreted as "A^* is provable in HA".
- M. : $\mathsf{iGLH} = \{A : \forall * \ (\mathsf{HA} \vdash A^*)\}$.
- $\mathsf{iGL} :=$
  - Axioms of Intuitionistic Logic.
  - $\mathsf{K} := \Box(A \to B) \to (\Box A \to \Box B)$.
  - $\mathsf{Lob} := \Box(\Box A \to A) \to \Box A$.
- $\mathsf{iGLH} := \mathsf{iGL}$ plus

$$\mathsf{H} := \{\Box A \to \Box B : A \mathrel{|\!\sim} B\}$$

- $\mathrel{|\!\sim}$ is a selection of admissible rules of iGL.
- Example: $(\neg A \to (B \vee C)) \mathrel{|\!\sim} ((\neg A \to B) \vee (\neg A \to C))$
- Example: $\neg\neg\Box A \mathrel{|\!\sim} \Box A$.

Attention.

Not all sound extensions of HA share the same PL.

PA is an obvious counterexample.

Extension by true Π_n-sentences

- $\Pi_0 := \Sigma_0 := \Delta_0$.
- $\Sigma_{n+1} := \exists\Pi_n := \{\exists x\, A : A \in \Pi_n\}$.
- $\Pi_{n+1} := \forall\Sigma_n := \{\forall x A : A \in \Sigma_n\}$.

- $\bar{\Pi}_0 := \bar{\Sigma}_0 := \Delta_0$.
- $\bar{\Sigma}_{n+1} := \exists\bar{\Pi}_n := \{\exists x\, A : A \in \bar{\Pi}_n\}$.
- $\bar{\Pi}_{n+1} := \forall(\bar{\Pi}_n \to \bar{\Sigma}_n) := \{\forall x(A \to B) : A \in \bar{\Pi}_n \;\&\; B \in \bar{\Sigma}_n\}$.

Definition.

- Let $\mathsf{HA}^n :=$ HA plus all true $\bar{\Pi}_n$-sentences.
- Let $\mathsf{PA}^n :=$ PA plus all true Π_n-sentences.

Theorem (F. Pakhomov & M.)

1. HA^n *has Disjunction Property:* $\mathsf{HA}^n \vdash A \vee B$ *implies either* $\mathsf{HA}^n \vdash A$ *or* $\mathsf{HA}^n \vdash B$.
2. HA^n *has Numerical Existence Property:* $\mathsf{HA}^n \vdash \exists x\ A(x)$ *implies* $\exists k \in \omega$ *such that* $\mathsf{HA}^n \vdash A(k)$.
3. HA^n *is* $\bar{\Sigma}^s_{n+1}$*-complete, i.e. for every true* $\bar{\Sigma}_{n+1}$*-sentence* A *we have* $\mathsf{HA}^n \vdash A$.
4. HA^n *is* $\mathsf{Bool}(\bar{\Pi}_n)$*-decidable.*
5. $\mathsf{PA}^n \vdash A$ *iff* $\mathsf{HA}^n \vdash A^{\neg}$.
6. HA^n *is closed under Markov Rule, i.e.* $\mathsf{HA}^n \vdash \neg\neg A$ *implies* $\mathsf{HA}^n \vdash A$ *for* $A \in \bar{\Sigma}_{n+1}$.
7. PA^n *is* $\bar{\Pi}_{n+2}$*-conservative extension of* HA^n.

Provability logic of HA^n

Theorem (F. Pakhomov & M.)

HA^n *has the same provability logic of* HA.

Proof idea. We first show that the Σ_{n+1}-PL of HA^n is same as Σ_1-PL of HA, say iGLH_σ. Then by the following result we are done:

$$\mathsf{iGLH} \nvdash A \;\Rightarrow\; \exists\theta\ \mathsf{iGLH}_\sigma \nvdash \theta(A) \;\Rightarrow\; \exists\sigma\ \mathsf{HA}^n \nvdash \sigma\theta(A)$$

Theorem (M. 2022)

iGLH *is the closure of* iGLH_σ *under substitutions.*

$$\mathsf{iGLH} \vdash A \quad \textit{iff} \quad \forall\theta\ (\mathsf{iGLH}_\sigma \vdash \theta(A))$$

Let $(\Box A)^*$ interpreted as "A^* is provable in T".

$$\mathsf{PL}(T, S) := \{A : \forall * \;\; S \vdash A^*\}$$

Observation.

$\mathsf{PL}(\mathsf{PA}^n, \mathsf{PA}) = \mathsf{PL}(\mathsf{PA}^n, \mathsf{PA}^n) = \mathsf{GL}.$

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Observation.

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Attension.

We only could prove $\mathsf{iGL} \subseteq \mathsf{PL}(\mathsf{HA}^n, \mathsf{HA}) \subseteq \mathsf{iGLH}$.

Question.

What is $\mathsf{PL}(\mathsf{HA}^n, \mathsf{HA})$?

HA^{n+}: an extension of HA^n

Define HA^{n+} as HA^n plus

$$\mathsf{PEM}(\bar{\Pi}_n) := \{A \vee \neg A : A \in \bar{\Pi}_n\}.$$

- $[n]_{\mathsf{i}}$ as HA^n-provability predicate.
- $[n]_{\mathsf{i}}^{+}$ as HA^{n+}-provability predicate.
- $[n]_{\mathsf{c}}$ as PA^n-provability predicate.

Observation.

Extensionally, HA^n and HA^{n+} are equal.

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Observation.

Extensionally, HA^n and HA^{n+} are equal.
Although this fact is PA and HA^n-verifiable, HA is not able to verify that for $n > 0$.

HA^{n+}-provability logic

Theorem (F. Pakhomov & M.)

$\mathsf{PL}(\mathsf{HA}^{n+}, \mathsf{HA}) = \mathsf{iGLH}.$

HA^{n+}-provability logic

Theorem (F. Pakhomov & M.)

$\mathsf{PL}(\mathsf{HA}^{n+}, \mathsf{HA}) = \mathsf{iGLH}$.

This theorem, encourage us to have a big step and consider the Intuitionistic Polymodal Provability Logic.

Classical Polymodal Provability Logic GLP

Language:

$\vee$, $\wedge$, $\to$, $\top$ and $[n]$ for $n \in \mathbb{N}$.

Polymodal PL:

Let $([n]A)^*$ interpreted as PA^n-provability of A^*.

$$\mathsf{PPL}(\mathsf{PA}) := \{A : \forall * \ \ \mathsf{PA} \vdash A^*\}$$

Theorem (G. Japaridze)

$\mathsf{PPL}(\mathsf{PA}) = \mathsf{GLP}$.

Axioms of GLP

- All axioms of Classical Logic
- $[n](A \to B) \to ([n]A \to [B])$.
- $[n]([n]A \to A) \to [n]A$.
- $[n]A \to [n+1]A$.
- $\neg[n]A \to [n+1]\neg[n]A$.
- $(A, A \to B)/B$.
- $A/[0]A$.

Intuitionistic Polymodal Provability Logic: first steps

Intuitionistic Polymodal PL:

Let $([n]A)^*$ interpreted as HA^{n+}-provability of A^*.

$$\mathsf{PPL(HA)} := \{A : \forall * \ \ \mathsf{HA} \vdash A^*\}$$

First candidate for PPL(HA)

$$\mathsf{iGLP} + \{[n]A \to [n]B : \mathsf{AR}_n(\mathsf{iGLP}, \mathrm{Boxed}_n) \vdash A \rhd B\}$$

iGLP

- All axioms of Intuitionistic Logic.
- $[n](A \to B) \to ([n]A \to [B])$.
- $[n]([n]A \to A) \to [n]A$.
- $[n]([i]A \vee \neg[i]A)$ for every $i < n$.
- $[n]A \to [n+1]A$.
- $A \to [n+1]A$ for every $A = [m]B \to \bigvee_i [n_i]B_i$ and $m \leq n$ and $n_i < n$.
- $(A, A \to B)/B$.
- $A/[0]A$.

$\mathsf{AR}_n(\mathsf{T}, \Delta)$

Ax : $A \rhd B$, for every $\mathsf{T} \vdash A \to B$.

Le_n: $A \rhd [n]A$ for every $A \in \mathcal{L}_{\mathsf{T}}$.

$\mathsf{V}(\Delta)$: $B \to C \rhd \bigvee_{i=1}^{n+m} B \xrightarrow{\Delta} E_i$, in which $B = \bigwedge_{i=1}^{n}(E_i \to F_i)$ and $C = \bigvee_{i=n+1}^{n+m} E_i$, and $A \xrightarrow{\Delta} B$ is a notation which is defined as follows:

$$A \xrightarrow{\Delta} B := \begin{cases} B & : B \in \Delta \\ A \to B & : \text{otherwise} \end{cases}$$

$$\frac{A \rhd B \qquad A \rhd C}{A \rhd B \wedge C}\ \text{Conj} \qquad \frac{A \rhd B \qquad B \rhd C}{A \rhd C}\ \text{Cut}$$

$$\frac{B \rhd A \qquad C \rhd A}{B \vee C \rhd A}\ \text{Disj} \qquad \frac{A \rhd B \qquad (C \in \Delta)}{C \to A \rhd C \to B}\ \text{Mont}(\Delta)$$

$$\Delta := \text{Boxed}_n := \{[i]A : i \leq n\} \cup \{\bot\}$$

$$T := \mathsf{iGLP}$$

Thanks For Your Attention