Searching for the ideal framework

Michael Rathjen

University of Leeds

Wormshop 2024

Gent, 4 September 2024

Proofs

> *A proof is a logical story that takes a reader from a place they know to a new, unvisited destination. [...] At times you arrive at what looks like an impasse and need to take a lateral step, moving sideways or even backwards to find a way around. Sometimes you need to wait for new mathematical characters [...] to be created so you can continue your journey.*

M. du Sautoy, What we cannot know, 2016.

Rafaello Bombelli 1526–1572

- Indian's in the 7^{th} century AD developed a theory of negative numbers.
- Negative numbers were banned in 13^{th} century Florence.
- 

Bombelli used a number i whose nature is that $i^2 = -1$.
- Descartes wrote about such numbers in his La Géométrie (1637) in which he coined the term imaginary and meant it to be derogatory (in the sense of ficticious).

> *Au reste tant les vrayes racines que les fausses ne sont pas tousjours reelles; mais quelquefois seulement imaginaires; c'est a dire qu'on peut bien tousjours en imaginer autant que jay dit en chasque Equation; mais qu'il n'y a quelquefois aucune quantité, qui corresponde a celles qu'on imagine [...]*

Leibniz on infinitesimals

> *Leibniz's own view, as published in 1689 and as repeated and elaborated subsequently in a number of letters, may be summarized as follows. While approving of the introduction of infinitely small and infinitely large quantities, Leibniz did not consider them as real, like the ordinary 'real' numbers, but thought of them as ideal or ficticious, rather like the imaginary numbers. However, by virtue of a general principle of continuity, these ideal numbers were supposed to be governed by the same laws as the ordinary numbers.*

A. Robinson, Metaphysics of the calculus (1967)

Hilbert: The method of ideal elements

- Solve a mathematical problem regarding a specific mathematical structure by adding new ideal elements to the structure.

Indispensable condition

- There is just one condition, albeit an absolutely necessary one, connected with the method of ideal elements. That condition is a **proof of consistency**, for the extension of a domain by the addition of ideal elements is legitimate only if the extension does not cause contradictions to appear in the old, narrower domain, or, in other words, only if the relations that obtain among the old structures when the ideal structures are deleted are always valid in the old domain. (Hilbert 1925)
- Another reading of Hilberts Programme:

  Elimination of ideal elements.

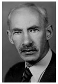
How much is needed?

- People say that **ZFC** is the gold standard.
- ,,*It is true that in the mathematics of today the higher levels of this hierarchy* [described by **ZFC**] *are practically never used. It is safe to say that 99.9% of present-day mathematics is contained in the first three levels of this hierarchy.*" Gödel (1951)

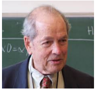
Minimalism

- Hermann Weyl, 1918 ,,Das Kontinuum".

- D. Hilbert, P. Bernays 1939: Fragments of Z_2 suffice.

- G. Takeuti, S. Feferman 1970–1995: Strength of **PA** suffices.

- **Reverse Math**: H. Friedman, S. Simpson, ... 1977–now.


G. Takeuti: A Conservative Extension of PA (1978)

We use higher type language. The use of higher type language is very convenient since it is the natural language for analysis and we can take all the definitions in analysis as they are without any change.

Finite Types

Definition of Finite Types: 0 is a finite type. If $\tau_1, \ldots, \tau_n$ are finite types, then

$$\tau = [\tau_1, \ldots, \tau_n]$$

is a finite type.

Definition An arithmetical formula is one that does not have higher type (> 0) quantifiers.

If $A(\alpha_1, \ldots, \alpha_n)$ is an arithmetical formula and $\alpha_1, \ldots, \alpha_n$ are of type $\tau_1, \ldots, \tau_n$, then

$$\{\varphi_1, \ldots, \varphi_n\}\, A(\varphi_1, \ldots, \varphi_n)$$

is an **abstract** of type $[\tau_1, \ldots, \tau_n]$.

If α is a free variable of type $[\tau_1, \ldots, \tau_n]$ and $V_1, \ldots, V_n$ are abstracts of type $\tau_1, \ldots, \tau_n$ resp., then $\alpha[V_1, \ldots, V_n]$ is a formula.

Quantifier Rules and Comprehension

$$\frac{F(V), \Gamma \Rightarrow \Delta}{\forall\varphi\, F(\varphi), \Gamma \Rightarrow \Delta}\ \forall\text{ left} \qquad \frac{\Gamma \Rightarrow \Delta, F(\alpha)}{\Gamma \Rightarrow \Delta, \forall\varphi\, F(\varphi)}\ \forall\text{ right}$$

$$\frac{F(\alpha), \Gamma \Rightarrow \Delta}{\exists\varphi\, F(\varphi), \Gamma \Rightarrow \Delta}\ \exists\text{ left} \qquad \frac{\Gamma \Rightarrow \Delta, F(V)}{\Gamma \Rightarrow \Delta, \exists\varphi\, F(\varphi)}\ \exists\text{ right}$$

α eigenvariable, V abstract (which is always arithmetic)

$$\forall\varphi\,(\varphi[0] \wedge \forall n(\varphi[n] \to \varphi[n+1]) \to \forall n\,\varphi[n])$$

There are also the standard axioms for the naturals arithmetic and for the rational numbers as a field as well as an axiom relating $\mathbb{N}$ and $\mathbb{Q}$.

What's the scope of Takeuti's theory FA?

- **FA** is conservative over **PA**. A cut elimination argument.
- The **reals** are introduced as Dedekind cuts of rationals.
- In **FA**, Takeuti develops real analysis, i.e., continuous functions, infinite series, differentiation, integrations, and also complex analysis up to what one does in a first course on complex analysis. Definition of Riemann's ζ-function via analytic continuation.
- What's missing? The Riemann mapping theorem, more general topological considerations, and the theory of Riemann surfaces seem to pose problems.

'Beautiful' theorem

Theorem: (Hadamard, de La Vallée Poussin 1896)

Prime number theorem

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln(x)}} = 1$$

where $\pi(x) =$ *number of prime numbers* $\leq x$.

Atle Selberg and Paul Erdös (1949) "elementary proof"

Mathematics in Explicit Mathematics

- S. Feferman: A language and axioms for explicit mathematics, 1975
- S. Feferman: Theories of finite type related to mathematical practice, 1977
- S. Feferman: Constructive theories of functions and classes, 1979
- S. Feferman: A theory of variable types, **VT** aka **VFT** , 1985

In **VFT**, Feferman develops 19^{th} c. analysis and much of 20^{th} c. analysis (without the coding of RM), concluding with the spectral theory for compact self-adjoint operators on a separable Hilbert space.

Theorem. **VFT** is conservative over **PA**.

Mathematical Conceptualism à la Weaver

Nick Weaver proposed a semi-intuitionistic theory **CM** of third-order arithmetic for axiomatizing what he calls mathematical conceptualism.

> The philosophical approach we adopt, mathematical conceptualism, is a refinement of the predicativist philosophy of Poincaré and Russell. The basic idea is that we accept as legitimate only those structures that can be constructed, but we allow constructions of transfinite length. What makes this "conceptual" is that we are concerned not only with those constructions that we can actually physically carry out, but more broadly with all those that are conceivable (perhaps supposing our universe had different properties than it does).

N. Weaver, Axiomatizing mathematical conceptualism in third order arithmetic. *arXiv:0905.1675v1*, 31 pages, 2009.

The system **CM**

1. **CM** has
   - first order variables $n, m, k, \ldots$ (thought of as ranging over $\mathbb{N}$)
   - second order variables $X, Y, Z, \ldots$ (thought of as ranging over sets of naturals)
   - third order variables $\mathbb{X}, \mathbb{Y}, \mathbb{Z}, \ldots$ (thought of as ranging over sets of sets of naturals)
   - **Axioms**
     1.1 Number-theoretic axioms
     1.2 Law of excluded middle for formulas with no second or third order quantifiers.
     1.3 Induction on naturals for all formulas.
     1.4 Dependent choice at the second order level:
     If $\forall n \forall X \exists Y\, \psi(n, X, Y)$ then
     $\forall X \exists Z\, [Z_{(0)} = X \wedge \forall n\, \psi(n, Z_{(n)}, Z_{(n+1)})]$.
     1.5 Comprehension:
     $\forall n(\varphi(n) \vee \neg\varphi(n)) \rightarrow \exists X \forall n\, [n \in X \leftrightarrow \varphi(n)]$
     $\forall X(\vartheta(X) \vee \neg\vartheta(X)) \rightarrow \exists \mathbb{Y} \forall X\, [X \in \mathbb{Y} \leftrightarrow \vartheta(X)]$

Borel, Baire, Lebesgues against the Axiom of Choice 1905

Borel: It seems to me that the objection against it is also valid for every reasoning where one assumes an arbitrary choice made an uncountable number of times, for **such reasoning does not belong in mathematics**.

Developing mathematics in **CM**

It's actually quite easy.

1. The reals are a third order object, inhabited by Dedekind cuts of rationals.
2. A topological space is a set $\mathbb{X}$ together with a family of subsets $\mathfrak{T}$ of $\mathbb{X}$ such that (i) $\emptyset$ and $\mathbb{X}$ belong to $\mathfrak{T}$; (ii) the union of any sequence of sets that belong to $\mathfrak{T}$ belongs to $\mathfrak{T}$; and (iii) the intersection of any finitely many sets that belong to $\mathfrak{T}$ belongs to $\mathfrak{T}$.
3. Weaver shows that lot of topology, measure theory and functional analysis can be developed in **CM**. Core mathematics can be straightforwardly implemented in **CM**.

What's the strength of **CM**?

Theorem (Shuwei Wang) **CM** has a realizability interpretation in Σ^1_1-DC, so its proof-theoretic ordinal is just $\varphi\varepsilon_0 0$.

Fundamental/Rudimentary Functions

For x an ordered pair (OP) $\langle a, b\rangle$ and y, z sets, define $1^{st}(x) = a$ and $2^{nd}(x) = b$ and

$y``\{z\} := \{u \mid \langle z, u\rangle \in y\}$.

The *fundamental functions* are as follows:

($\mathcal{F}_p$) $\mathcal{F}_p(x, y) := \{x, y\}$,

($\mathcal{F}_\cap$) $\mathcal{F}_\cap(x, y) := x \cap \bigcap y$

($\mathcal{F}_\cup$) $\mathcal{F}_\cup(x, y) := \bigcup x$,

($\mathcal{F}_\setminus$) $\mathcal{F}_\setminus(x, y) := x \setminus y$,

($\mathcal{F}_\times$) $\mathcal{F}_\times(x, y) := x \times y$,

($\mathcal{F}_\rightarrow$) $\mathcal{F}_\rightarrow(x, y) := x \cap \{z \mid \mathrm{OP}(y) \wedge (z \in 1^{st}(y) \rightarrow z \in 2^{nd}(y))\}$,

($\mathcal{F}_\forall$) $\mathcal{F}_\forall(x, y) := \{x``\{z\} \mid z \in y\} = \{\{u \mid \langle z, u\rangle \in x\} \mid z \in y\}$,

($\mathcal{F}_d$) $\mathcal{F}_d(x, y) := \mathrm{dom}(x) = \{1^{st}(z) \mid z \in x \wedge z \textit{ is an ordered pair}\}$,

($\mathcal{F}_r$) $\mathcal{F}_r(x, y) := \mathrm{ran}(x) = \{2^{nd}(z) \mid z \in x \wedge z \textit{ is an ordered pair}\}$,

($\mathcal{F}_{123}$) $\mathcal{F}_{123}(x, y) := \{\langle u, v, w\rangle \mid \langle u, v\rangle \in x \wedge w \in y\}$,

($\mathcal{F}_{132}$) $\mathcal{F}_{132}(x, y) := \{\langle u, w, v\rangle \mid \langle u, v\rangle \in x \wedge w \in y\}$,

($\mathcal{F}_=$) $\mathcal{F}_=(x, y) := \{\langle v, u\rangle \in y \times x \mid u = v\}$,

($\mathcal{F}_\in$) $\mathcal{F}_\in(x, y) := \{\langle v, u\rangle \in y \times x \mid u \in v\}$.

Weaver, Analysis in J_2

Definition. The rudimentary closure of a set x is the smallest set y such that $x \subseteq y$, $x \in y$, and y is closed under application of the fundamental functions.

Definition. $J_0 = \emptyset$; J_1 is the rudimentary closure of J_0; J_2 is the rudimentary closure of J_1.

Definition. An ι-set is an element of J_2.
An ι-class is a definable subset of J_2 whose intersection with every ι-set is an ι-set.
Weaver then develops mathematics in J_2, where the reals are an ι-class. He shows that quite a chunk of mathematics, including central topics of topology, measure theory, and Banach spaces can be accounted for in J_2.

J_2 can be turned into a classical set theory, say T_{J_2}, so that mathematics can be carried out formally in T_{J_2}.

Theorem (R., Wang) The theory T_{J_2} has proof-theoretic strength $\varepsilon_{\varepsilon_0}$ (same as ACA).

Fermat's Last Theorem

For $n > 2$, the equation

$$x^n + y^n = z^n$$

has no non-trivial solution in $\mathbb{N}$ (or $\mathbb{Q}$).

Proved by Andrew Wiles in 1995. He proved enough of the **Taniyama-Shimura conjecture** to deduce *FLT*.

The proof of *FLT* builds on a massive amount of abstract mathematics, developed by Grothendieck and others.

Large workspaces

Given one space X, Grothendieck took an array of categories of related spaces and sheaves on them as one simply and explicitly organized workspace guiding proofs about X.

He would "approach these categories from a 'naïve' point of view, as if we were dealing with sets" [Grothendieck & Dieudonne, 1971]. Grothendieck aimed to preserve what he liked calling the

childish ... incorrigible naïveté

of his geometry.

"... to avoid certain logical difficulties, we will accept the notion of a Universe, which is a set 'large enough' that the habitual operations of set theory do not go outside of it [Grothendieck, '71]

There are means to show the great cohomological proofs like Deligne ['74], or Faltings ['83], or Wiles ['95] never need to go beyond **ZFC**.

Frameworks for Constructive Mathematics (in the 1970s)

- S. Feferman, Explicit Mathematics
- J. Myhill, Constructive Set Theory, **CST**.
- P. Martin-Löf, Intuitionistic Type Theory, **MLTT**.
- P. Aczel, Constructive Zermelo-Fraenkel Set Theory, **CZF**.

  CZF is a simplification and extension of Myhill's **CST**, induced by **MLTT**.

Constructive Zermelo-Fraenkel set theory, **CZF**

- **Extensionality**
- **Pairing**, **Union**, **Infinity**
- **Bounded Separation**
- **Subset Collection**

  For all sets A, B there exists a "sufficiently large" set of multi-valued functions from A to B.

- **Strong Collection**

  $$(\forall x \in a)\, \exists y\; \varphi(x, y) \;\rightarrow$$
  $$\exists b\, [\, (\forall x \in a)\, (\exists y \in b)\; \varphi(x, y) \;\wedge\; (\forall y \in b)\, (\exists x \in a)\; \varphi(x, y)\,]$$

- **Set Induction scheme**

  http://www1.maths.leeds.ac.uk/ rathjen/book.pdf

Principles of Omniscience

Limited Principle of Omniscience (**LPO**):

$$\forall f \in 2^{\mathbb{N}}\, [\exists n\, f(n) = 1 \quad \vee \quad \forall n\, f(n) = 0].$$

Lesser Limited Principle of Omniscience (**LLPO**):

$$\forall f \in 2^{\mathbb{N}} \big(\forall n, m[f(n) = f(m) = 1 \rightarrow n = m] \\ \rightarrow\ [\forall n\, f(2n) = 0 \quad \vee \quad \forall n\, f(2n+1) = 0]\big).$$

Dummett "Thought and Reality" 2006

- "If there are no gaps in reality, that is no questions that have no answers, then God's logic will be classical.

  Those many people who favour classical over intuitionistic logic are therefore guilty of the presumption of reasoning as if they were God."

  Bertrand Russell: ".... it's just a medical condition ..."

Bishop's critique of Brouwer

[t]he movement Brouwer founded has long been dead, killed partly by compromises of Brouwer's disciples with the viewpoint of idealism, partly by extraneous peculiarities of Brouwer's system which made it vague and even ridiculous to practising mathematicians, but chiefly by the failure of Brouwer and his followers to convince the mathematical public that abandonment of the idealistic viewpoint would not sterilize or cripple the development of mathematics. (1967)

CM and Constructive Zermelo-Fraenkel set theory (**CZF**)

It is shown that it is unexpectedly easy to formalize a great deal of modern functional analysis in **CM**.

The interesting connection between $\mathbf{CZF} + \mathbf{LPO} + \mathbf{RDC}$ and **CM** is the following.

Theorem. **CM** can be interpreted in $\mathbf{CZF} + \mathbf{LPO} + \mathbf{RDC}$.

Theorem (R. 2013) $\mathbf{CZF} + \mathbf{LPO} + \mathbf{RDC}$ and **CZF** have the same proof-theoretic strength.

Paul Lorenzen: Konstruktive Begründung der Mathematik (1950)

Obwohl es gerechtfertigt wäre, axiomatische Mengenlehren, für die kein konstruktives Modell vorhanden ist, einfach als mißglückte Versuche in Zukunft beiseite zu lassen, besteht aber auch die andere Möglichkeit, zu untersuchen, ob sich solche Axiomatisierungen – nachdem sie einmal da sind und wir uns dran gewöhnt haben – nicht doch auf irgend einem Umweg als kalkulatorisch zweckmässige Fiktionen für die konstruktive Mathematik verwenden lassen.
[...]
Im Gegensatz zu den intuitionistischen Versuchen darf dabei jetzt – nach einwandfreier Begründung der Logik – das tertium non datur stets benutzt werden. Die so unbequeme Beschränkung auf ,,entscheidbare" Eigenschaften, ,,berechenbare reelle Zahlen" usw. ist nicht mehr erforderlich.

CZF with large sets

Laura Crosilla, R.: Inaccessible set axioms may have little consistency strength Annals of Pure and Applied Logic (2002).

Monotone inductive definitions in $\mathbf{T}_0$

- In Explicit Mathematics, $\mathbf{T}_0$, one can easily formalize an axiom, **MID**, asserting that every monotone operation on classifications has a least fixed point.

  What is the strength of $\mathbf{T}_0 + \mathbf{MID}$? [...] I have tried, but did not succeed, to extend my interpretation of $\mathbf{T}_0$ in Σ^1_2-$AC + BI$ to include the statement **MID**. The theory $\mathbf{T}_0 + \mathbf{MID}$ includes all constructive formulations of iteration of monotone inductive definitions of which I am aware, while $\mathbf{T}_0$ (in its *IG* axiom) is based squarely on the general iteration of accessibility inductive definitions. Thus it would be of great interest for the present subject to settle the relationship between these theories. (p. 88)

  S. Feferman, 1982, Monotone inductive definitions.

Pushing the envelope

Theorem 1 (Glaß, R., Schlüter) For all Σ^1_2 sentences φ:

$$\mathsf{KPi}^r + \exists\gamma\,(L_\gamma \prec_1 L) \vdash \varphi \quad\Leftrightarrow\quad \Sigma^1_2\text{-}\mathsf{AC}_0 + \Pi^1_2\text{-}\mathsf{CA}^- \vdash \varphi$$
$$\Leftrightarrow\quad T_0{\restriction} + \mathsf{MID}$$

$$\mathsf{KPi}^w + \exists\gamma\,(L_\gamma \prec_1 L) \vdash \varphi \quad\Leftrightarrow\quad \Sigma^1_2\text{-}\mathsf{AC} + \Pi^1_2\text{-}\mathsf{CA}^- \vdash \varphi$$
$$\Leftrightarrow\quad T_0{\restriction} + \mathsf{IND}_{\mathbb{N}} + \mathsf{MID}$$

Theorem 2 (R.) The following have the same proof-theoretic strength:

1. Π^1_2-$\mathbf{CA}_0$ and $\mathbf{T}_0 \restriction + \mathsf{UMID}_{\mathbb{N}}$.
2. Π^1_2-$\mathbf{CA}$ and $\mathbf{T}_0 \restriction + \mathsf{IND}_{\mathbb{N}} + \mathsf{UMID}_{\mathbb{N}}$.

Theorem 3 (Tupailo) Π^1_2-$\mathbf{CA}_0$ and $\mathbf{T}^i_0 \restriction + \mathsf{UMID}_{\mathbb{N}}$ have the same proof-theoretic strength.

Theorem 4 (Tupailo, R.) Π^1_2-$\mathbf{CA}$ and $\mathbf{T}^i_0 \restriction + \mathsf{IND}_{\mathbb{N}} + \mathsf{UMID}_{\mathbb{N}}$ have the same proof-theoretic strength.

- S. Takahashi, 1989, *Monotone inductive definitions in a constructive theory of functions and classes*.
- R., 1996, *Monotone inductive definitions in explicit mathematics*.
- T. Glaß, M. Rathjen, A. Schlüter, 1997, *The strength of monotone inductive definitions in explicit mathematics*.
- R., 1998, *Explicit Mathematics with the monotone fixed point principle*.
- R., 1999, *Explicit Mathematics with the monotone fixed point principle. II: Models*.
- R., 2002, *Explicit Mathematics with monotone inductive definitions: a survey.*
- S. Tupailo, 2004, *On the intuitionistic strength of monotone inductive definitions.*
- R., S. Tupailo, 2004, *On the strength of UMID in intuitionistic explicit mathematics*.