Possible worlds and the contingency of logic

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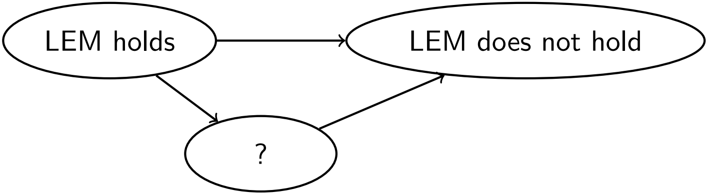
Regular modal logic semantics

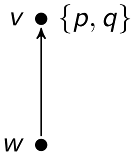
- Necessitation: If a statement can be proved, then it is necessarily true everywhere.
- Axioms and theorems of a logical system are true in every possible world considered by that system.
- The traditional view of a singular logic accurately representing all possible worlds and their behaviour has been challenged. Could we create a new modal logical system to better reflect this?

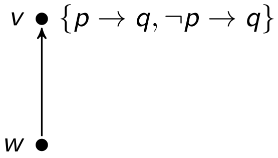
Possible worlds with contingent logic



- We focus on a first example, combining classical and intuitionistic reasoning.

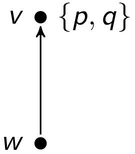
Local reasoning





- w behaves classically (as in K)
- $x \Vdash \Box A \iff \forall y(xRy \Rightarrow y \Vdash A)$

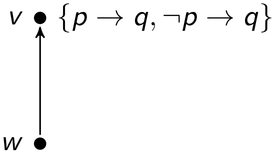
Local reasoning



$w \Vdash \Box p \to \Box p$

$w \Vdash \Box p \to \Box q$

$w \Vdash \Box (p \to q)$



$w \Vdash \Box q$

Local versus non-local reasoning

Desired properties of the models

- Kripke frame **F** with a set T of formulas assigned to each node, such that T is:
  - Consistent, i.e. $\bot \notin T$;
  - Closed under classical/intuitionistic local reasoning;
  - $\Box$ behaves classically with respect to the frame (as in K).

Language

- Language $\mathcal{L}_\Box := p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A$
- Set $\mathrm{Form}_\Box$ of formulas in $\mathcal{L}_\Box$

Defining local reasoning

- A *local classical derivation* $\mathcal{D}$ from Γ to φ ($\Gamma, \varphi \subseteq \text{Form}_\Box$) is a sequence of formulas $\varphi_1, \varphi_2, ..., \varphi_k$ s.t $\forall i \in \{1, 2, ..., k\}$:
  - $\varphi_i \in \Gamma$ or
  - φ_i is in the form of a Classical axiom (CPC axioms) in the language $\mathcal{L}_\Box$ or
  - There is $j, l < i$ such that φ_j is of the form $\varphi_l \to \varphi_i$
  - $\varphi_k = \varphi$.

  We write $\Gamma \vdash_{\mathsf{c}}^{\mathcal{L}_\Box} \varphi$.
- When $\Gamma \vdash_{\mathsf{c}}^{\mathcal{L}_\Box} \varphi$ we say we can locally deduce (in a classical world) φ from Γ.
- This reasoning does not use the rule of Necessitation or the Distribution axiom $\Box(A \to B) \to (\Box A \to \Box B)$ present in the classical modal logic K.

Defining the Language and Derivations

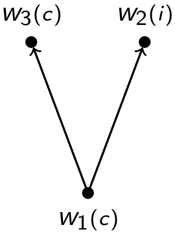
- We similarly define the local intuitionistic derivations.
- $\overline{\mathsf{T}}^{c\Box}/\overline{\mathsf{T}}^{i\Box}$ is the closure of T over $\vdash_{\mathsf{c}}^{\mathcal{L}\Box}$ / $\vdash_{\mathsf{i}}^{\mathcal{L}\Box}$.

Mixed models

- A *mixed model* is a tuple $\mathcal{M} := \langle W, R, e \rangle$ where $\langle W, R \rangle$ is a Kripke frame and e is an *extension* $e : W \to \mathcal{P}(\mathrm{Form}_\Box) \times \{i, c\}$ $\big(\text{denoted } e(w) = \langle T_w, l_w \rangle\big)$ such that:
  1. $\bot \notin T_w$;
  2. $T_w \vdash^{\mathcal{L}\Box}_{\mathrm{l_w}} \varphi \Rightarrow \varphi \in T_w$ (i.e. closure under local deduction);
  3. $\Box\varphi \in T_w \iff \forall v(wRv \Rightarrow \varphi \in T_w)$;
  4. $\neg\Box\varphi \in T_w \iff \exists u(wRu \wedge \varphi \notin T_u)$.

$$T_{w_2} = \overline{\{p, q\} \cup \{\Box\varphi \mid \varphi \in \mathbf{Form}_\Box\}}^{i_\Box}$$

$$T_{w_1} = \overline{\{\neg q\} \cup \{\Box\varphi \mid \varphi \in T_{w_2}\} \cup \{\neg\Box\psi \mid \psi \in \mathbf{Form}_\Box / T_{w_2}\}}^{c_\Box}$$



$$T_{w_3} = \overline{\{p\} \cup \{\Box\varphi \mid \varphi \in \mathbf{Form}_\Box\}}^{c_\Box}$$

$$T_{w_2} = \overline{\{p, q\} \cup \{\Box\varphi \mid \varphi \in \mathbf{Form}_\Box\}}^{i_\Box}$$

$$T_{w_1} = \overline{\{\neg p \vee q\} \cup \{\Box\varphi \mid \varphi \in T_{w_2} \cap T_{w_3}\}}$$
$$\overline{\cup\{\neg\Box\psi \mid \psi \in \mathbf{Form}_\Box / T_{w_2} \cap T_{w_3}\}}^{c_\Box}$$

$w_1(c)$ ⇄ $w_2(i)$

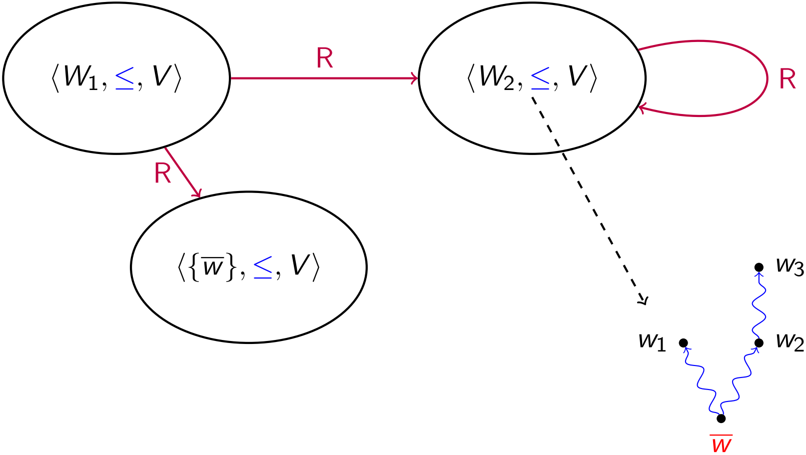
Concrete models

- Concrete Model $\mathcal{M} := \langle \mathbf{F}, \lambda, m \rangle$ From a Kripke frame $F = \langle W, R \rangle$ and function $\lambda : W \to \{c, i\}$, we assign to each $w \in W$ a rooted intuitionistic Kripke model $m(w) := \langle U_w, \leq_w, V_w \rangle$ (root: $\overline{w} \in U_w$) s.t. $\lambda(w) = c \Rightarrow U_w = \{\overline{w}\}$
- $\Vdash$ is defined on $\Theta := \bigcup_{w \in W} U_w$

  (For $x \in U_w$:)

  1. $x \nVdash \bot$ and $x \Vdash \top$;
  2. $x \Vdash p$ iff $x \in V_w(p)$;
  3. $x \Vdash A \wedge B$ iff $x \Vdash A$ and $x \Vdash B$;
  4. $x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$;
  5. $x \Vdash A \to B$ iff $\forall y \in U_w (x \leq y \to y \nVdash A$ or $y \Vdash B)$;
  6. $x \Vdash \neg A$ iff $x \Vdash A \to \bot$;
  7. $x \Vdash \Box A$ iff $\forall v \in W (wRv \to \overline{v} \Vdash A)$.

Concrete models



Concrete models to mixed models

Theorem

From a concrete model $\mathcal{M}$ we can obtain a mixed model $\mathcal{M}'$ such that

$$\mathcal{M}, w \Vdash \varphi \iff \mathcal{M}', w \Vdash \varphi$$

- Example of a non-concrete mixed model.
  - $F = \langle \{w\}, R \rangle$, $R = \varnothing$, $I_w = c$;
  - $T_w = \overline{\{p \vee q\} \cup \{\Box\varphi \mid \varphi \in \mathbf{Form}_\Box\}}^c$

Soundness for $\mathcal{MM}$

- We define the logic *MixL* $:= iK + \Box A \vee \neg \Box A$

Theorem

Soundness: The logic MixL is sound with respect to the class $\mathcal{MM}$ of all mixed models.

- Results of interest:
  - (Necessitation)$M \vDash A$ implies $M \vDash \Box A$
  - (Distributivity)$M \vDash \Box(A \to B) \to (\Box A \to \Box B)$
  - (Box excluded middle)$M \vDash \Box A \vee \neg \Box A$

Quick proof of Distributivity (k-axiom)

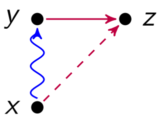
- We show $\Box(A \to B) \to (\Box A \to \Box B) \in T_w$:
  - $(\mathbb{A}\ \Box(A \to B) \in T_w)$
    - If $\Box A \in T_w$, $\forall y \in M(A, A \to B \in T_y \Rightarrow B \in T_y) \Rightarrow \Box B \in T_w \Rightarrow \Box(A \to B) \to (\Box A \to \Box B) \in T_w$
    - If $\Box A \notin T_w$, $\Box A \to \bot \in T_w$, and by reductio ad absurdum, $\Box A \to \Box B \in T_w \Rightarrow \Box(A \to B) \to (\Box A \to \Box B) \in T_w$
  - $((\mathbb{A}\ \Box(A \to B) \notin T_w)$, then $\Box(A \to B) \to \bot \in T_w$ and by reductio ad absurdum, $\Box(A \to B) \to (\Box A \to \Box B) \in T_w$

Intuitionistic logic and Modal logic: Semantics

- Kripke semantics for IPC:
  - $M = (W, \leq, V)$ (Monotonicity w.r.t. V)
  - $M, w \Vdash A \to B$ iff for all $v \geq w$: $M, v \Vdash A$ implies $M, v \Vdash B$
- Possible world semantics for K:
  - $M = (W, R, V)$
  - $M, w \Vdash \Box A$ iff for all v s.t. wRv: $M, v \Vdash A$

Birelational semantics for iK

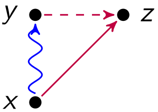
- $M = (W, \leq, R, V)$ (Monotonicity w.r.t. V)
  - $M, w \Vdash A \to B$ iff for all $v \geq w$: $M, v \Vdash A$ implies $M, v \Vdash B$
  - $M, w \Vdash \Box A$ iff for all v s.t. wRv: $M, v \Vdash A$
- Frame property (F0):



$$x \leq y \Rightarrow (yRz \Rightarrow xRz)$$

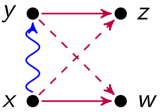
Mixed birelational models

- Frame condition for $\Box A \vee \neg \Box A$ (F3):



$$x \leq y \Rightarrow (yRz \Leftarrow xRz)$$

- *Mixed birelational model* $M = (W, \leq, R, V)$:
  - Monotonicity w.r.t. V
  - Frame property (F0+F3):



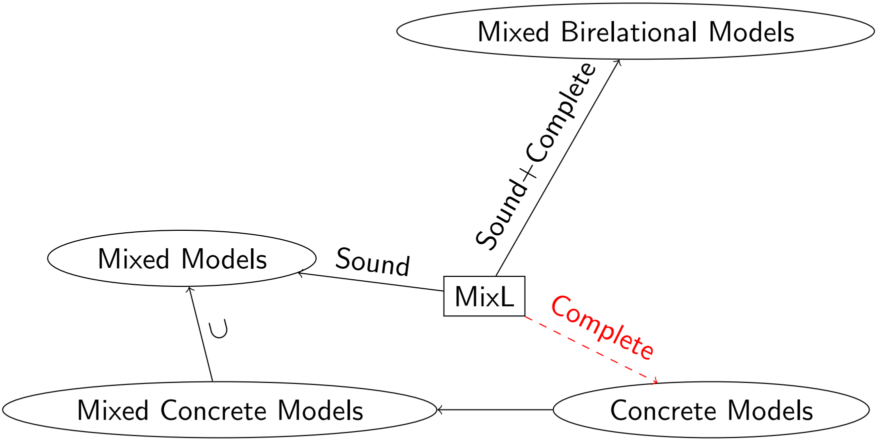
$$x \leq y \Rightarrow (yRz \iff xRz)$$

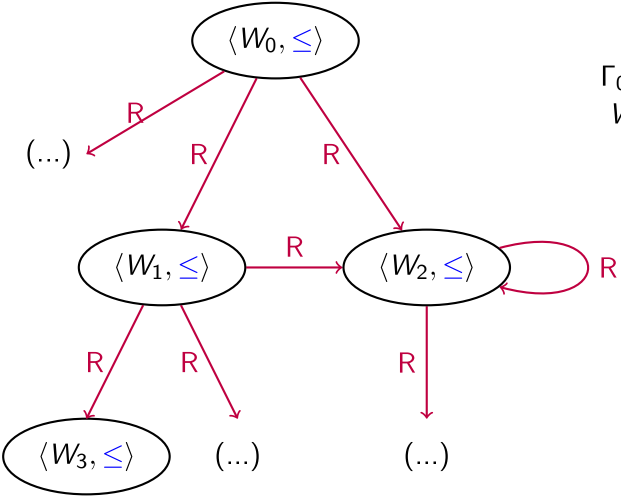
Mixed birelational models

Theorem

MixL is sound and complete with respect to Mixed birelational models

- Proof method: Henkin-style canonical model construction:
  - Prime sets Γ:
    - $\bot \notin \Gamma$;
    - Closed under MixL;
    - $\varphi \vee \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$.
  - $\mathcal{M} := \langle W, R, \leq, V \rangle$
    - $W := \{\Gamma \mid \Gamma \text{ is a prime set}\}$;
    - $\Gamma \leq \Delta :\iff \Gamma \subseteq \Delta$;
    - $\Gamma R \Delta$ if and only if ($\Box\varphi \in \Gamma$ implies $\varphi \in \Delta$);
    - $p \in V(\Gamma) :\iff p \in \Gamma$.


Mixed Birelational Models
Sound+Complete
Mixed Models
Sound
MixL
Complete
∪
Mixed Concrete Models
Concrete Models



Γ_0 counterprime of φ

$W_i = \{\Delta \mid \Gamma_i \leq \Delta\}$

$\forall i \ (\Gamma_0 R^* \Gamma_i)$

Theorem

MixL is sound and complete with respect to concrete models.

Corollary

MixL is sound and complete with respect to mixed models.

What next?

- **Conjecture**
  The class $\mathcal{CM}$ of all concrete models gives the class of all mixed models such that for all $w \in M$, T_w is a prime theory ($\varphi \vee \psi \in T_w \Rightarrow \varphi \in T_w$ or $\psi \in T_w$).
- Finite model property.
- Including $\Diamond$ in the semantical definition.
- Can possibly include more/different logics in this framework:
  - Incomparable logics
  - Many valued
  - Etc.