Admissibility of Visser's Rules in Intuitionistic Modal Logics

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Outline

1. Motivation
   - Universal proof theory
   - Computational content of proofs
2. Constructive Axioms
3. The Constructive Base
4. Main theorem

Universal proof theory

A recent project investigating the generic behavior of proof systems.

Aim

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We address the following problems:

- the *existence problem*: investigates the existence of proof systems of a given form
- the *equivalence problem*: focuses on natural equivalence relations among these systems.

Existence problem

So far, the focus has been on the existence problem.

Main idea (method of invariants)

existence of a proof system of a certain form for a logic L

$$\Longrightarrow$$

a pure logical property for L

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Hence,

the absence of this property

$$\Longrightarrow$$

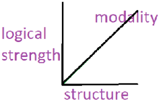
non-existence of proof systems of the given form

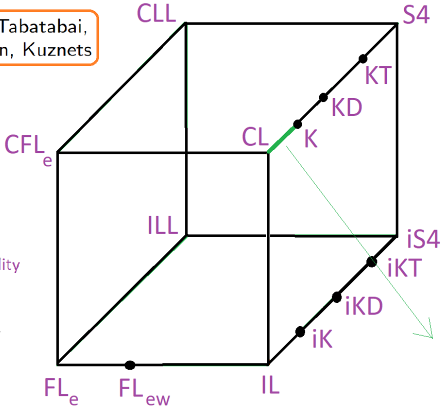
By choosing a rare property (e.g. interpolation) we get stronger results.

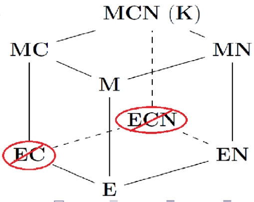
Universal Proof Theory (a team work)

Class of logics	Restricted proof system	Property
cube	semi-analytic	CIP, LIP
a subset of cube	terminating semi-analytic	UIP, ULIP
intuitionistic modal	constructive	disjunction property
intuitionistic modal	constructive	admissible Visser rules

with Iemhoff, Tabatabai, van der Giessen, Kuznets







Computational content of proofs

Proofs in constructive mathematics have more information than provability.

Common mathematical practice:

Now (theorem)	**Later** (meta-theorem)
Forget the information and only talk about provability.	Talk about the construction and the hidden information in the proof.

Aim: Identifying *constructive* proof systems.
To guarantee that the proofs are constructive, and although we forget the information now, we can extract it later.

Extracting information

Example (Extracting information)

- From a constructive proof of $\exists x A(x)$, we can obtain a witness t and a proof of $A(t)$.
  - For instance in **LJ**, using cut elimination: $$\frac{\Rightarrow A(t)}{\Rightarrow \exists x A(x)}$$
- From a constructive proof of $A \vee B$, we can obtain either a proof for A or for B (Disjunction Property, DP).

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How do we extract the information?

- Proofs are usually complicated (e.g. they contain cuts in sequent calculi or redundant parts in natural deduction).
- After simplifying the proofs (e.g. cut elimination, normalization), we can extract information.

But (designing a proof system allowing for) simplifying proofs is not always possible! Even if it is, it will be very costly.

Example (Related work)

DP in intuitionistic propositional logic, IPC, can be witnessed in p-time:

- (Buss, Mints '99) used natural deduction system (via normalization).
- (Buss, Pudlák '01) used sequent calculus (via cut elimination).

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Usually to prove the admissibility of an admissible rule[1] in a logic, one needs some sort of cut elimination.
Is it possible to extract information in a way that avoids cut elimination?

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Our goal

Yes! As a non-trivial setting, we choose the modal language:

- Present a precise syntactic form for constructively acceptable axioms.
- Provide a feasible extraction algorithm for the theories axiomatized by constructively acceptable axioms over a reasonable constructive base.

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Note that:

- We are not extracting information from the proofs in one *specific* system but a *general* family of calculi, only by knowing the form of the axioms.
- The extraction process is feasible.
- No need for any sort of cut elimination.

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Constructive Axioms

Constructive axioms, whatever they are, must be careful with the positive occurrences of disjunctions.

The following are not constructively acceptable:

- $\neg r \vee \neg\neg r$
- $\neg(p \wedge q) \to \neg p \vee \neg q$ as it proves $\neg r \vee \neg\neg r$. Note that the former is assumed as an axiom.

How to define constructive axioms?

A first proposal for the propositional language:

Avoid all positive occurrences of disjunction!

There are two problems with this proposal:

- It is too strict and rejects even some constructively accepted formulas such as the axioms $p \to p \vee q$ and $p \wedge (q \vee r) \to (p \wedge q) \vee (p \wedge r)$.
- Allows indirect introduction of positive disjunctions through nested implications, e.g., $\neg\neg p \to p$.

Our proposal

- Allow only "depth[2] two nested implications". The problematic formulas such as $\neg\neg p \to p$ have depth three or more.
- The real problem is not the disjunctions but the way that they are mixed with implications, e.g., $p \vee \neg p$ and $(p \to q) \vee (q \to p)$.

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- The real problem is not the disjunctions but the way that they are mixed with implications, e.g., $p \vee \neg p$ and $(p \to q) \vee (q \to p)$. Therefore, define constructive formulas as:

Start with depth two formulas with no positive occurrences of disjunction and then substitute the atoms by implication-free formulas.

For instance, we can save:

- $p \to (p \vee q)$
- $p \wedge (q \vee r) \to (p \wedge q) \vee (p \wedge r)$

both as an implication-free substitution of depth one formula $s \to t$.

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Constructive Modal Axioms

Treating $\Diamond$ **as a disjunction** and $\Box$ **as an implication**, one can extend the above proposal to the modal language. More precisely, set $\mathcal{L} = \{\wedge, \vee, \to, \bot, \top, \Box, \Diamond\}$. Then:

Definition

- Basic: $\{\wedge, \vee, \Diamond\}$ over atoms (including $\top$ and $\bot$). (substituters)
- Almost positive: $\{\wedge, \vee, \Box, \Diamond\}$ over basics and $A \to B$, where A is basic and B is almost positive. (depth one)
- Constructive: $\{\wedge, \Box\}$ over basics and $A \to B$, where A is almost positive and B is constructive.

Some Examples

Example

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almost positive	$\neg p,\ (p \vee \neg p),\ \Diamond^m \Box^n p,\ \Box^m \Diamond^n p$	$\neg\neg p,\ (p \to q) \to r,\ \Box p \to q$

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The formulas

$$((p \to q) \to r) \to s \qquad (\Box p \to q) \to r$$

are neither almost positive nor constructive.

Example

The formula $(\mathbf{ga_{klmn}})$:

$$\Diamond^k \Box^l p \to \Box^m \Diamond^n p$$

is constructive and covers all the following modal axioms:

- $(\mathbf{T_a}) : \Box p \to p$
- $(\mathbf{T_b}) : p \to \Diamond p$
- $(\mathbf{B_a}) : \Diamond \Box p \to p$
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For more complicated examples:

- ($\Box\!\to$) : $(\Diamond p \to \Box q) \to \Box(p \to q)$
- ($\mathbf{.2}$) : $\Diamond(p \wedge \Box q) \to \Box(p \vee \Diamond q)$
- ($\mathbf{bw_n}$) : $\bigwedge_{i=0}^{n} \Diamond p_i \to \bigvee_{0 \leqslant i \neq j}^{n} \Diamond(p_i \wedge (p_j \vee \Diamond p_j))$

Some modal axioms

name	axiom	name	axiom
K_a	$\Box(p \to q) \to (\Box p \to \Box q)$	K_b	$\Box(p \to q) \to (\Diamond p \to \Diamond q)$
$\neg\Diamond\bot$	$\neg\Diamond\bot$	$\Diamond\vee$	$\Diamond(p \vee q) \to \Diamond p \vee \Diamond q$
$\Box \to$	$(\Diamond p \to \Box q) \to \Box(p \to q)$	ga	$\Diamond\Box p \to \Box\Diamond p$
4_a	$\Box p \to \Box\Box p$	4_b	$\Diamond\Diamond p \to \Diamond p$
B_a	$\Diamond\Box p \to p$	B_b	$p \to \Box\Diamond p$
5_a	$\Diamond\Box p \to \Box p$	5_b	$\Diamond p \to \Box\Diamond p$
c_a	$p \to \Box p$	c_b	$\Diamond p \to p$
$4_{n,m,a}$	$\Box^n p \to \Box^m p$	$4_{n,m,b}$	$\Diamond^m p \to \Diamond^n p$
$den_{r,a}$	$\Box^{r+1} p \to \Box^r p$	$den_{r,b}$	$\Diamond^r p \to \Diamond^{r+1} p$
$tra_{n,a}$	$\bigwedge_{i=1}^{n} \Box^i p \to \Box^{n+1} p$	$tra_{n,b}$	$\Diamond^{n+1} p \to \bigvee_{i=0}^{n} \Diamond^i p$
ga_{klmn}	$\Diamond^k\Box^l p \to \Box^m\Diamond^n p$	d_1	$\neg\Diamond p \to \Box\neg p$
d_2	$\Box\neg p \to \neg\Diamond p$	d_3	$\Diamond\neg p \to \neg\Box p$
bw_r	$\bigwedge_{i=0}^{r} \Diamond p_i \to \bigvee_{0\leqslant i\neq j}^{r} \Diamond(p_i \wedge (p_j \vee \Diamond p_j))$	H	$p \to \Box(\Diamond p \to p)$
$M_{\Diamond}^{\to}$	$(p \to q) \to (\Diamond p \to \Diamond q)$	dir	$\Diamond(\Box p \wedge q) \to \Box(\Diamond p \vee q)$
bd_r	defined in the caption	$.2$	$\Diamond(p \wedge \Box q) \to \Box(p \vee \Diamond q)$

Table 1: Some modal axioms. Everywhere $k, l, m, n \geqslant 0$ and $r \geqslant 1$. In $4_{n,m,a}$ and $4_{n,m,b}$ we assume $0 \leqslant n < m$ and in ga_{klmn}, we assume either $k \geqslant 1$ or $m \geqslant 1$. The formula bd_r is defined recursively: $bd_1 = \neg p_0 \to \Box\neg\Box p_0$ and $bd_{r+1} = \neg p_r \to \Box(\Box p_r \to bd_r)$.

Outline

The Constructive Base

The sequent calculus **CK** is **LJ** (including cut) plus the following rules:

$$\frac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\,(K_{\Box}) \qquad \frac{\Gamma, A \Rightarrow B}{\Box\Gamma, \Diamond A \Rightarrow \Diamond B}\,(K_{\Diamond})$$

Sometimes the base is considered **IK**: extension of **CK** by the axioms:

$$\neg\Diamond\bot \qquad \Diamond(p \vee q) \to \Diamond p \vee \Diamond q \qquad (\Diamond p \to \Box q) \to \Box(p \to q)$$

We can extend **CK** or **IK** by some axioms, e.g.,

- T: $\Box p \to p$ and $p \to \Diamond p$,
- 4: $\Box p \to \Box\Box p$ and $\Diamond\Diamond p \to \Diamond p$,
- 5: $\Diamond p \to \Box\Diamond p$ and $\Diamond\Box p \to \Box p$,
- B: $\Diamond\Box p \to p$ and $p \to \Box\Diamond p$.

A Technical Definition

Definition

The calculus $G = \mathbf{CK} + \mathcal{A}$ is

- T-free if it is valid in the irreflexive Kripke frame of one node.
- T-full if it is valid in the reflexive Kripke frame of one node and proves $\Box p \to p$ and $p \to \Diamond p$.

Example

Let $\mathcal{A}$ be a set of axioms in Table 1. Then,
$\mathbf{CK} + \mathcal{A}$ is T-free and $\mathbf{CK} + \mathcal{A} \cup \{T_a, T_b\}$ is T-full.
However, the system $\mathbf{CK} + \neg\Box\bot$ is neither T-free nor T-full.

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Main theorem

Theorem

Let $G = \mathbf{CK} + \mathcal{A}$ be a T-free or a T-full calculus, where $\mathcal{A}$ is a set of constructive axioms. Then, G has the feasible disjunction property.
It means that there is a polynomial time algorithm that reads a G-proof of $A \vee B$ and outputs either a G-proof for A or a G-proof for B.

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More generally, feasible admissibility of all Visser's rules also holds, i.e., there is a polynomial time algorithm that reads a G-proof of

$$\{A_i \to B_i\}_{i \in I} \Rightarrow C \vee D$$

and outputs a G-proof for one of the following sequents:

$$\{A_i \to B_i\}_{i \in I} \Rightarrow C \qquad \{A_i \to B_i\}_{i \in I} \Rightarrow D \qquad \{A_i \to B_i\}_{i \in I} \Rightarrow A_j,$$

for some $j \in I$.

Remarks

- Feasibility is sensitive to the proof system. As we allow cut, all natural axiomatizations are feasibly equivalent. Hence, one can use the theorem for Hilbert-style or natural deduction systems.
- Even forgetting feasibility, the result is strong as it proves the disjunction property without any need for any good proof system with cut elimination. One can simply present the logic by its axioms!
- The result can be adopted to the fragments lacking one or both modalities.

Positive Applications

Corollary

Let $\mathcal{A}$ be a finite set of axioms in Table 1. Then, the sequent calculi $\mathbf{CK} + \mathcal{A}$ and $\mathbf{CK} + \mathcal{A} \cup \{T_a, T_b\}$ have the feasible disjunction property.

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Corollary

The calculi $\mathbf{CK}X$ and $\mathbf{IK}X$, for any $X \subseteq \{T, B, 4, 5\}$, including **CS4**, **CS5**, **IS4**, and **IS5** (also known as **MIPC**), have feasible disjunction property.

We proved the feasible disjunction property uniformly for a family of logics, only judging by the syntactic form of the axioms in $\mathcal{A}$.

Negative Applications

Theorem (Iemhoff)

IPC is the only intermediate logic that admits all Visser's rules.

Therefore,

Corollary

IPC is the only intermediate logic that is axiomatizable by a set of constructive axioms over **LJ**.

Corollary

Let $L \neq$ IPC be an intermediate logic and $\mathcal{A}$ be a set of axioms in Table 1. Then, none of the logics $L\mathrm{CK} + \mathcal{A}$ and $L\mathrm{CK} + \mathcal{A} \cup \{T_a, T_b\}$ are axiomatizable by a set of constructive axioms over **CK**.

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- For any $A \in \mathcal{L}$ add $\langle A \rangle$ as a new atom to $\mathcal{L}$. The new language: $\mathcal{L}^+$.

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- For any $A \in \mathcal{L}$ add $\langle A \rangle$ as a new atom to $\mathcal{L}$. The new language: $\mathcal{L}^+$.
- Define a natural translation function $t : \mathcal{L} \to \mathcal{L}^+$:
  - $p^t = \langle p \rangle$, for any atom p;
  - $(A \circ B)^t = (A^t \circ B^t) \wedge \langle A \circ B \rangle$, for any $\circ \in \{\wedge, \vee, \to\}$.
- Clearly $\vdash A^t \Rightarrow \langle A \rangle$.

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- There is a standard substitution $s : \mathcal{L}^+ \to \mathcal{L}$, replacing $\langle A \rangle$ by A.
- Horn formula: a formula in the form $\bigwedge_{i \in I} p_i \to q$.
- Unit propagation: If some Horn formulas prove a disjunction between two atoms (even classically), they prove one of them intuitiontically.

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Thank you for your attention.