How constructive is Gödel's Dialectica translation?

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Overview of the talk

- Gödel's Dialectica translation is an interpretation of HA in a quantifier-free theory T of primitive recursive functionals of finite type.
- Gödel claimed that T is more constructive than HA, because it is quantifier-free.
- **Circularity objection** (Kreisel, Troelstra, Tait, Ferreira): T is not quantifier-free; rather, quantificational logic is secretly presupposed in defining the relevant class of functionals.
- Gödel denied all charges, but no one understood his response.
- I'll try to defend Gödel—but the constructive foundations of T are really weird!

Outline

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Gödel: doubts about the constructivity of HA

The essence of constructivity is avoiding non-constructive existence proofs. Does HA succeed at this?

- HA interprets PA *too easily*.
- Non-constructive $\exists$ theorems of PA translate easily into $\neg\forall\neg$ theorems of HA. Isn't that suspicious? Maybe intuitionistic logic contains some non-constructive elements?
- HA can prove $\neg\forall x\, A(x)$ without exhibiting any specific counterexample!

+ further doubts about the BHK explanation (BHK provability is essentially unformalizable)

Gödel: T is more constructive than HA

- Best way to avoid non-constructive ∃ proofs is to get rid of quantifiers entirely.
- T is more constructive than HA because it is *quantifier-free* and has decidable primitives.

> *Let me call a system strictly constructive or finitistic if it satisfies these three requirements (relations and functions decidable, respectively, calculable, no existential quantifiers at all, and no propositional operations applied to universal propositions). (Gödel 1941)*

Gödel: foundational significance of the D-translation

Gödel argues that the Dialectica translation gives us. . .

- a genuinely *constructive meaning* for the basic notions of intuitionistic logic (in HA),
- a *constructive consistency proof* for arithmetic,
- a better understanding of the sense in which HA *is* constructive.

Outline

Troelstra's objection

Troelstra writes:

> *Gödel's aim was to replace the abstract intuitionistic logical notions by a notion of functional, as concrete as possible; he succeeded in fact in eliminating the logic except for the logic hidden in the precise definition of the intended class of functionals. . . . In short, there is some reductive gain, though it is not clear-cut; we think it falls short of Gödel's aims. (Troelstra 1990)*

Tait's objection

Tait writes:

> *[Gödel believed] that a constructive theory of functions of finite type requires that the higher type variables range only over functions which are provably computable (berechenbaren). But he never spelled out a satisfactory account of this which avoids the very intuitionistic logic that he was attempting to reinterpret. (Tait 2006)*

N.B.: "computable" here does not mean Turing-computable.

Gödel's definition of "computable function of finite type"

The concept "computable function of type t" is defined as follows: 1. The computable functions of type 0 are the natural numbers. 2. If the concepts "computable function of type t_0*", "computable function of type* t_1*", ..., "computable function of type* t_k*" (where* $k \geq 1$*), have already been defined, then a computable function of type* $(t_0, t_1, \ldots, t_k)$ *is defined to be a well-defined mathematical procedure which can be applied to any k-tuple of computable functions of types* $t_1, \ldots, t_k$*, and yields a computable function of type* t_0 *as result; and for which, moreover, this general fact is constructively evident. The phrase "well-defined mathematical procedure" is to be accepted as having a clear meaning without any further explanation. (Gödel 1972)*

Simplified definition of "computable function of finite type"

f is a computable function of type $\alpha \to \beta$ if and only if:

1. f is a well-defined mathematical procedure that can be applied to any object of type α to yield an object of type β, and
2. it is constructively evident that the first clause holds.

Circularity objection, v.1

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Circularity objection (v.1): this "any" leads to increasing quantificational complexity as you pass to higher and higher types.

Circularity objection, v. 1

- Let's drop the second clause ("constructively evident") for now.
- Try unpacking the definition of "f is a comp fn of type $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$". We get:

  f is a well-defined mathematical procedure that takes every well-defined mathematical procedure that takes every number to a number, to a number.

$$\forall x(\forall y(y : \mathbb{N} \supset x(y) : \mathbb{N}) \supset f(x) : \mathbb{N}).$$

Circularity objection, v. 1

- But now we have propositional operations being applied to universal quantifiers.
- The quantificational complexity only increases as we pass to higher and higher types.
- Does it help to restore the second clause?

$$\forall x(\forall y(y : \mathbb{N} \supset x(y) : \mathbb{N}) \supset f(x) : \mathbb{N})$$

Circularity objection, v. 2

Gödel says "constructively evident or demonstrable"; later he says "reductively provable." What does this mean?

- Troelstra and Tait: maybe reductive provability has to do with Tait computability predicates (strong normalization of T)?
- But these have unbounded quantificational complexity as we pass to higher and higher types.
- Gödel also talks about HRO. Similar problem.

Circularity objection (v.2): reductive provability must involve some hidden quantificational reasoning.

Outline

The basic idea of reductive proofs

- Gödel defines reductive proof as follows: "up to certain trivial supplementations, the *chain of definitions* of the concepts occurring in the theorem . . . forms by itself a proof."
- Reductive proof is based on Leibniz's notion of *analyticity*.
- Reductive proofs mainly involve reasoning about definitions and the type character of what is defined.
- How does this help with circularity?

Reductive provability is a factive modality

- Think of reductive provability as a modality, $\Box$.
- Assume $\Box$ is *factive*, i.e., $\Box\varphi \supset \varphi$.
- Then the nestings of $\forall$ and $\supset$ are "shielded" at every step:

$$\Box\forall x(\Box[\forall y(y : \mathbb{N} \supset x(y) : \mathbb{N})] \supset f(x) : \mathbb{N}).$$

Without factivity, we would have $\varphi \wedge \Box\varphi$ and we would still have unshielded nestings of $\forall$ and $\supset$.

Reductive provability is a decidable modality

For, also the statement "$(x)\varphi(x)$ is reductively provable" . . . means that a certain procedure of checking the chain of definitions of the concepts in φ yields a certain result. (draft of Gödel 1972)

. . . because [reductive] proofs are uniquely determined by the theorems, quantifications over "any proof" can be avoided. (Gödel 1972)

Questions about reductive provability

- What are the means of proof allowed in a reductive proof?
- Do reductive proofs involve quantificational reasoning?
- What is the decision procedure for $\Box\varphi$?
- Note that $\Box$ occurs within the scope of $\Box$. What's going on here?

Good news: we can make the notion of reductive provability completely precise.

(But there are still some conceptual issues to be worked out.)

What can occur in a reductive proof?

- definitions of combinators K, S, Z and recursors R, e.g.,

$$x : \alpha \supset y : \beta \supset K_{\alpha\beta} xy = x$$

- definitions of the types,

$$t : \alpha \to \beta \leftrightarrow \Box(x : \alpha \supset t(x) : \beta)$$

- axioms for prop. logic, $=$, successor
- axioms and rules for reductive provability:

  1. $\Box\varphi \to \varphi$
  2. from a reductive proof of φ, infer $\Box\varphi$
  3. from $\Box(x : \alpha \supset \varphi)$, infer $x : \alpha \supset \Box\varphi$

- restricted rules of modus ponens, substitution, induction.

Motivation for principles (1)–(3)

1. $\Box\varphi \to \varphi$
   Proofs establish the truth of their conclusions.

2. from a reductive proof of φ, infer $\Box\varphi$
   Proofs should be recognizable as proofs.

   The combination of (1) and (2) usually leads to Montague's Paradox. But not in this setting, because the proof of the Diagonal Lemma is not a *reductive* proof.

3. from $\Box(x : \alpha \supset \varphi)$, infer $x : \alpha \supset \Box\varphi$
   ???

So, is T strictly constructive?

On my reading . . .

- T is quantifier-free.
- T involves reductive provability, $\Box$. But this modality is decidable, so it's not so bad.
- Reductive proofs involve reasoning about reductive provability itself. Is that constructively OK?

Closing remarks

- Does reductive provability connect to the normalization of terms? Or to the ordinal analysis of T?
- Reductive provability has a striking combination of features: it is factive and self-reflexive, yet consistent and decidable!
- Are there other interesting factive notions of provability?

For Further Reading I

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  Revised and expanded English translation of (Gödel 1958). First published in *Kurt Gödel: Collected Works, Vol. II*, eds. Feferman et al, Oxford University Press, 1990.
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For Further Reading III

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